Sums of powers in function fields in one variable over $\mathbb{R}$

In 1934 E. Witt showed that every sum of squares in such a field is in fact a sum of two squares. In the 1960's it was A. Pfister who broadly extended this result to function fields in n variables over any real closed field. In this latter case, 2^n squares suffice. Regarding a generalization to the case of sums of powers with exponents larger than 2 much less is known. This is partly due to the fact that the well developed theory of quadratic forms has no counterpart in the setting of higher degree forms. New methods have to be found.

The present talk will focus on the one-variable case over $\mathbb{R}$ where a blend of algebraic and topological ideas have already led to interesting and somewhat satisfactory results. After briefly touching non-formally-real function fields the talk will turn to a formally-real function field F in one variable over $\mathbb{R}$. Such field is the function field of a smooth projective curve over $\mathbb{R}$. The set of real points γ of this projective curve turns out to be a compact 1-dimensional C^∞-manifold, hence diffeomorphic to the topological sum of finitely many, say r, circle S^1. Furthermore, each element $f \in F$ induces a continuous function $\hat{f}$ on γ with values in the real projective line $\mathbb{P}^1$. The first main result reads:

the representation $F \to C(\gamma, \mathbb{P}^1), f \mapsto \hat{f}$ has dense image

relative to the compact-open topology.

This result finds an algebraic interpretation which in turn is one key for quantitative results on sums of higher powers. Let $H := \{f \in F \mid f(\gamma) \subseteq \mathbb{R}\}$. It turns out that this ring, usually referred to as the real holomorphy ring of F, is a Dedekind ring with quotient field F of finite class number 2^r. The structure of its group of units and the ideal structure of H is a rich source for statements on sums of powers of any exponent. In particular, the representation theorem above is seen to be equivalent to the following surprising fact:

every totally positive unit of H is the sum of 2 squares of totally positive units.

This statement can be understood as a variant of Witt's result quoted above. So far, no other proof has been found that avoids the topological representation. Using all this one can prove:

(1) given an exponent n there exists a higher Pythagoras number p_n such that every sum of n- th powers in F is a sum of at most p_n terms of nth powers,
(2) there are feasible bounds for p_n,
(3) $p_3 \leq 3, p_4 \leq 6$.