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Existencia y Multiplicidad de soluciones para ciertos Problemas Elípticos.

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Contents

Introduction

In this work we will study the existence of bounded positive solutions for some elliptic system. In general, this type of system has the form

$$\begin{cases} -\Delta u + V_1(x)u = f(x,u,v) & \text{in} \quad \Omega \\ -\Delta v + V_2(x)v = g(x,u,v) & \text{in} \quad \Omega \end{cases} \tag{1}$$

where Ω is some domain in $\mathbb{R}^N$, $f, g : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ are caraethedory functions and where V_i is a potential satisfying some properties. Let us just mention two classes of variational systems that have been object of much research in these last decades, a gradient system and a Hamiltonian system (see [19]). More precisely,

a) the System (1) is said to be *gradient*, if there exists a differentiable function $G : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ such that
$$\frac{\partial G}{\partial u} = f \quad \text{and} \quad \frac{\partial G}{\partial v} = g,$$

b) and the System (1) is said to be *Hamiltonian*, if there exists a differentiable function $F : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ such that
$$\frac{\partial F}{\partial u} = g \quad \text{and} \quad \frac{\partial F}{\partial v} = f.$$

The variational terminology comes from the fact that in both case, System (1) has a naturally associated functional with the system. However, if the System (1) is not variational, we may use another methods such as:

The moving planes method. In this method is essentially restricted to the case $f = f(v)$ and $g = g(u)$, with f, g nondecreasing. Furthermore, either f, g have to satisfy some technical assumption or Ω be convex, see [16]. This method was first introduced in [21] for the scalar case.

Blow-up method. This method proceeds by contradiction, by assuming that (1) do not have a priori bounds for the positive solutions. This method was introduced by Gidas and Spruk in [23] for scalar problems. Later, it was successfully extended to many types of systems, for instance see [22, 44].

Hardy-Sobolev inequalities. This method is based on using the first eigenfunction of the Laplacian as a multiplier to derive an estimate on the nonlinear terms. Under proper growth assumptions on f, g, this estimate is then improved on an H^1 bound by using Hardy-Sobolev inequalities, and then into a uniform bound using some bootstrap arguments. This method was introduced by Brezis and Turner in [10] for scalar problems and then extended to certain classes of systems in [13, 15, 16].

Lower and upper solution method. This method, can be applied, for instance, when the nonlinearities have a sublinear growth near zero. For instance, see [35].

The main objective of this work is to establish the existence of two bounded solutions for a certain, not necessarily variational, system defined in $\mathbb{R}^N$. The first solution will be obtained by the lower an upper method and the second solution will be obtained considering an auxiliary problem, which is variational. More precisely, we will study the following elliptic system involving Schrödinger operators

$$\begin{cases} -\Delta u + V_1(x)u = \lambda \rho_1(x)(u+1)^r(v+1)^p & \text{in} \quad \mathbb{R}^N \\ -\Delta v + V_2(x)v = \mu \rho_2(x)(u+1)^q(v+1)^s & \text{in} \quad \mathbb{R}^N, \\ u(x), v(x) \to 0 & \text{as} \quad |x| \to \infty. \end{cases} \tag{$S_{\lambda,\mu}$}$$

where $\lambda, \mu > 0$, $p, q, r, s \geq 0$, $N \geq 3$ and V_i is a nonnegative vanish potential satisfying

$$\frac{a_i}{1+|x|^\alpha} \leq V_i(x) \leq \frac{A_i}{1+|x|^\alpha} \quad \text{for all} \quad x \in \mathbb{R}^N \tag{H_V^α}$$

for some constants $\alpha, A_i > 0$ and $a_i \geq 0$, $i = 1, 2$. The weight $\rho_i \in L^\infty(\mathbb{R}^N)$ satisfies

$$0 < \rho_i(x) \leq \frac{k_i}{1+|x|^\beta} \quad \text{in} \quad \mathbb{R}^N, \tag{H_ρ}$$

with $\alpha + \beta > 4$ and $k_i > 0$, $i = 1, 2$.
Notice that this type of system such as $(S_{\lambda,\mu})$ appears when we are looking for stationary waves solutions of the following coupled nonlinear Schrödinger equations

$$\begin{cases} i\phi_t + \Delta\phi - (V_1(x) + \omega_1)\phi + \lambda\rho_1(x)(u+1)^r(v+1)^p = 0 \\ i\psi_t + \Delta\psi + (V_2(x) + \omega_2)\psi + \mu\rho_2(x)(u+1)^q(v+1)^s = 0, \\ \phi(t,x), \psi(t,x) : [0,\infty) \times \mathbb{R}^N \to \mathbb{C} \end{cases} \tag{2}$$

where the solution is given by

$$(\phi, \psi) = (e^{i\omega_1 t}u(x), e^{i\omega_2 t}v(x)), \ \omega_1, \omega_2 \in \mathbb{R}^+. \tag{3}$$

This kind of Schrödinger system can be used to describe many physical phenomena, such as the propagation of pulses in optical fiber [33] and a binary mixture of Bose-Einstein condensates [18]. The mathematical studies of stationary wave solutions for Schrödinger equations has attracted much attention since 1970s, see e.g. [6, 40]. Using (3), it is easy to see that getting a stationary wave solution of (2) is equivalent to solving the elliptic system $(S_{\lambda,\mu})$ for $(u(x), v(x))$.

Before dealing with the main results of System $(S_{\lambda,\mu})$, we will give some know facts about the scalar equation

$$-\Delta u = \rho(x) \text{ in } \mathbb{R}^N. \tag{4}$$

We will say that ρ has the so called property (H), introduced by Brezis and Kamin [9], if the equation (4) has a bounded solution. In **Chapter 2** we prove that the sublinear problem

$$-\Delta u = \rho(x)u^q \ \text{ in } \ \mathbb{R}^N, \quad N \geq 3 \tag{5}$$

where $0 < q < 1$, has a bounded positive solution if and only if ρ has the property (H). This result is due to the celebrated paper [9] by Brezis and Kamin. An important result, which we present in the same section, is that Problem (5) has a bounded solution if and only if

$$U(x) = \frac{c}{|x|^{N-2}} * \rho \in L^\infty(\mathbb{R}^N). \tag{6}$$

Moreover, in **Proposition 2.1.13**, we will show that when ρ is a radially symmetric function, then for all $x \in \mathbb{R}^N$ with $|x| = r$,

$$U(x) = U(r) = \int_r^{+\infty} \left(s^{1-N} \int_0^s t^{N-1} \rho(t) dt \right) ds,$$

which allows us to show that **(6)** is satisfied if we consider potentials like

$$\rho(x) = \frac{1}{1 + |x|^\beta}, \text{ for any } \beta > 2.$$

In this same way, we will study the existence of bounded solution for the *linear Schrödinger equation*

$$-\Delta u + V(x)u = \rho(x) \text{ in } \mathbb{R}^N. \tag{LS}$$

giving a condition of "compatibility" between ρ and V. As we will see in **Section 2.2** the compatibility condition tells us that ρ and the product VU has the property (H) where U is a bounded solution of **(4)**. For instance, we will show that V and ρ are compatible when the potential V satisfies (H_V^α) and ρ satisfies (H_ρ) with $\alpha \in (0, 2]$. So, using an argument like the one exposed in [9], we guarantee the existence of bounded solutions for **(LS)** that vanishes into infinity. This result is due to recent work by Cardoso, Cerda, Pereira and Ubilla [11].
Among other classes of Schrödinger equations, they also considered the following scalar equation

$$\begin{cases} -\Delta u + V(x)u = \lambda\rho(x)(u+1)^p & \text{in} \quad \mathbb{R}^N \\ u(x) \to 0 & \text{as} \quad |x| \to \infty \end{cases} \tag{$P_{\lambda,p}$}$$

where $1 < p < (N + 2)/(N - 2)$. Under the conditions (H_V^α) and (H_ρ), they showed the existence of a bounded positive solution of Problem $(P_{\lambda,p})$ and also a second solution, via variational methods, for sufficiently small $\lambda > 0$.

In **Chapter 3**, we will give our main results related to elliptic system $(\mathbf{S}_{\lambda,\mu})$. In **Section 3.1** assuming the conditions (H_V^α), (H_ρ) with $\alpha \in (0, 2]$ and using upper and lower solution technique, we first prove the existence of a bounded positive solution of System $(\mathbf{S}_{\lambda,\mu})$. We observe that in [9], as well as in [11], the existence and uniqueness of solution in bounded domains was crucial to get an increasing sequence of solutions in balls that converge, as the radius goes to infinity, to the solution of the original problem in whole $\mathbb{R}^N$. As far as we know, the first work for elliptic systems using the ideas of [9], was done by Montenegro [35], where uniqueness of solution in balls also plays an important role. Since System $(\mathbf{S}_{\lambda,\mu})$ in bounded domains does not have this property, we will have to use an alternative argument which involves minimal solutions.

Let us state our first result.

Theorem 1. *Assume that $p, q, r, s \geq 0$ and in addition suppose hypotheses (H_ρ) and (H_V^α) hold with $\alpha \in (0, 2]$ and $\alpha + \beta > 4$. Then, there exists $\Lambda > 0$ such that System $(\mathbf{S}_{\lambda,\mu})$ has at least one bounded positive solution for every $0 < \lambda, \mu < \Lambda$.*

We can also establish a converse for the previous theorem:

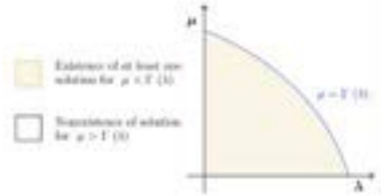
Theorem 2. *Suppose that $V \in L^\infty(\mathbb{R}^N)$ is a nonnegative potential and the weights ρ_i belong to $L^\infty(\mathbb{R}^N)$ with $\rho_i > 0$, for $i = 1, 2$. Suppose also that $\lambda, \mu > 0$, the powers satisfy $0 < r, s < 1$, $pq < (r-1)(s-1)$ and there exist positive constants b_1, b_2 such that $b_1\rho_1(x) \leq \rho_2(x) \leq b_2\rho_1(x)$ for every $x \in \mathbb{R}^N$. If System $(\mathbf{S}_{\lambda,\mu})$ admits a bounded positive solution, then, the linear Schrödinger equation **(LS)** has a bounded positive solution when $\rho = \rho_1$ as well as when $\rho = \rho_2$.*

Note that when $r, s > 1$ we can construct a function that is the border between the region of existence and nonexistence.

Theorem 3. *Suppose hypotheses (H_ρ) and (H_V^α) hold with $\alpha \in (0, 2]$ and $\alpha + \beta > 4$. Assume also that $r, s > 1$ and $p, q \geq 0$. Then, there is a positive constant λ^* and a continuous function $\Gamma : (0, \lambda^*) \to [0, \infty)$ such that if $\lambda \in (0, \lambda^*)$ then System $(\mathbf{S}_{\lambda,\mu})$:*

i) has at least one bounded positive solution if $0 < \mu < \Gamma(\lambda)$;

ii) has no bounded positive solution if $\mu > \Gamma(\lambda)$.



It is worth noting that to obtain existence results of positive bounded solutions of System $(\mathbf{S}_{\lambda,\mu})$ it is essential to impose the decay hypotheses on the weight $\rho(x)$. In fact, note that when $\rho_i(x) = 1$ and $V_i(x) = 0$, System $(\mathbf{S}_{\lambda,\mu})$ is given by

$$\begin{cases} -\Delta u = \lambda(u+1)^r(v+1)^p & \text{in} \quad \mathbb{R}^N \\ -\Delta v = \mu(u+1)^q(v+1)^s & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{7}$$

thus the (H_ρ) property is not satisfied, hence we cannot apply our results. Moreover, there are no solution of (7) in the following cases (see [36, Theorem 3]):
$p, q, r, s \geq 0$, $p + r \leq q + s$ and

$$\begin{cases} p + r \leq p_s & \text{if} \quad r > 1 \text{ or } q + s = p + r \\ s \leq p_s & \text{if} \quad r \leq 1,\ q = 0 \text{ and } q + s > p + r \\ b > \dfrac{2}{p_s - 1} & \text{if} \quad r \leq 1,\ q > 0,\ q + s > p + r \text{ and } pq > (r-1)(s-1), \end{cases}$$

where

$$b = \frac{2(q + 1 - r)}{pq - (1 - r)(1 - s)} \quad \text{and} \quad p_s = \frac{N}{N - 2}$$

is the so called Serrin's exponent.

On the other hand, the second solution will be obtained employing variational methods. Here we will consider two types of systems. The first one is the following gradient system

$$\begin{cases} -\Delta u + V(x)u = \lambda\rho_1(x)(u+1)^r(v+1)^{s+1} & \text{in} \quad \mathbb{R}^N \\ -\Delta v + V(x)v = \lambda\rho_2(x)(u+1)^{r+1}(v+1)^s & \text{in} \quad \mathbb{R}^N, \\ u(x), v(x) \to 0 & \text{as} \quad |x| \to \infty \end{cases} \tag{$\mathbf{GS}_\lambda$}$$

with $r, s > 1$, $r + s < 2^* - 2$, $\rho_1(x) = (r+1)\rho(x)$ and $\rho_2(x) = (s+1)\rho(x)$. To obtain this second solution we will use the Mountain Pass Theorem [43, Theorem 1.17]. The main result in this context is the following:

Theorem 4. *Suppose hypotheses (H_ρ) and (H_V^α) hold with $\alpha \in (0, 2]$ and $\alpha + \beta > 4$,*

i) If $r, s \geq 0$, then there exists $\lambda^* > 0$ such that the gradient System (GS_λ) *possesses at least one bounded positive solution $(u_{1,\lambda}, v_{1,\lambda})$ for all $0 < \lambda < \lambda^*$ while for $r, s > 1$ and $\lambda > \lambda^*$ there are no bounded positive solutions.*

ii) If $r, s > 1$ and $r + s < 2^* - 2$, then there exists $0 < \lambda^{**} \leq \lambda^*$ such that the gradient System (GS_λ) *possesses a second positive solution of the form $(u_{1,\lambda} + u, v_{1,\lambda} + v)$ for all $0 < \lambda < \lambda^{**}$, where $u, v \in H^1(\mathbb{R}^N)$.*

The second situation involves the following Hamiltonian system

$$\begin{cases} -\Delta u + V(x)u = \lambda\rho(x)(v+1)^p & \text{in} \quad \mathbb{R}^N \\ -\Delta v + V(x)v = \lambda\rho(x)(u+1)^q & \text{in} \quad \mathbb{R}^N, \\ u(x), v(x) \to 0 & \text{as} \quad |x| \to \infty \end{cases} \tag{HS_λ}$$

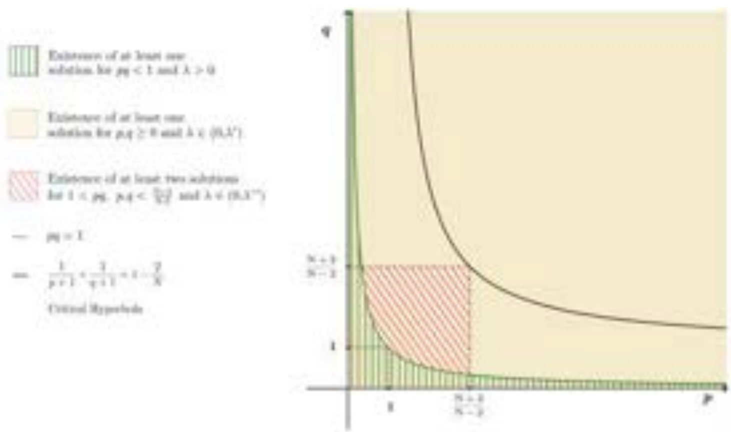
for some conditions in the powers $p, q > 0$. In this case, in order to obtain the existence of a second solution we will use a linking theorem proved in [31, Theorem 2.1]. The main result involving the Hamiltonian system is the following:

Theorem 5. *Suppose hypotheses (H_ρ) and (H_V^α) hold with $\alpha \in (0, 2]$. Also, suppose also that $\alpha + \beta > 4$ and $p, q \geq 0$, then*

i) There exists $\lambda^* > 0$ such that Hamiltonian System (HS_λ) *possesses at least one bounded positive solution $(u_{1,\lambda}, v_{1,\lambda})$ for all $0 < \lambda < \lambda^*$ while for $p, q > 1$ and $\lambda > \lambda^*$ there are no bounded positive solutions.*

ii) If $pq < 1$, then Hamiltonian System (HS_λ) *possesses at least one bounded positive solution $(u_{1,\lambda}, v_{1,\lambda})$ for all $\lambda > 0$.*

iii) If $1 < pq$ and $p, q < 2^* - 1$, then there exists $0 < \lambda^{**} \leq \lambda^*$ such that Hamiltonian System (HS_λ) *possesses a second positive solution of the form $(u_{1,\lambda} + u, v_{1,\lambda} + v)$ for all $0 < \lambda < \lambda^{**}$, where $u, v \in H^1(\mathbb{R}^N)$.*



This graph illustrates the results obtained for System (HS_λ), which may be compared to works about Hamiltonian systems involving the critical hyperbola.

We would like to point out that in both **Theorem 4** and **Theorem 5**, to show existence of a second solution we will use an auxiliary problem which allow us to avoid imposing additional hypotheses of integrabilities on the weights ρ_i. Here, the potential V_i also plays an important role in defining adequate spaces in which we will consider the associated energy functional.

In **Section 3.4**, we give an application of **Theorem 1**. For this purpose, let us introduce the following System, which, in part, has motivated our study:

$$\begin{cases} -\Delta z = \rho_1(x) z^r w^p & \text{in} \quad \mathbb{R}^N \\ -\Delta w = \rho_2(x) z^q w^s & \text{in} \quad \mathbb{R}^N, \\ z(x) \to c_1, w(x) \to c_2 & \text{as} \quad |x| \to \infty \end{cases} \tag{8}$$

where ρ_i satisfies (H_ρ) with $\beta > 2$ and $c_1, c_2 > 0$. Note that the solutions of this System do not belong to any Sobolev space, so it is difficult to solve directly. However, as we will see in the last section, a strategy involving **Theorem 1** allows us to find a solution of System **(8)**, which apparently is the only way to solve it.

To conclude this work, in **Chapter 4** we will study the Poisson's equation in the half space:

$$-\Delta u = \rho(x) \text{ in } \mathbb{R}^N_+, \tag{9}$$

where

$$\mathbb{R}^N_+ = \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_N > 0\},$$

and $\rho \in L^\infty_{loc}(\mathbb{R}^N_+)$, $\rho(x) \geq 0$ and ρ not identically zero. For this purpose notice that if $y = (y_1, \dots, y_N) \in \mathbb{R}^N_+$, its reflection in the plane $\partial\mathbb{R}^N_+$, is the point

$$\tilde{y} = (y_1, \dots, -y_N),$$

then Green's function in $\mathbb{R}^N_+$ is given by

$$G_{\mathbb{R}^N_+}(x, y) = \Gamma(x - y) - \Gamma(x - \tilde{y}) \quad \text{for all} \quad x \neq y \quad \text{in} \quad \mathbb{R}^N_+,$$

where Γ is the fundamental solution of the Laplace equation.
Recently, in [1] using some characterization of the Green's function $G_{\mathbb{R}^N_+}$, show that

$$v(x) = \int_{\mathbb{R}^N_+} G_{\mathbb{R}^N_+}(x, y)\rho(y)dy \quad \text{for } x \in \mathbb{R}^N,$$

is a solution of

$$\begin{cases} -\Delta u = \rho(x) & \text{in} \quad \mathbb{R}^N_+ \\ u = 0 & \text{on} \quad \mathbb{R}^N \backslash \mathbb{R}^N_+ \end{cases} \tag{10}$$

for every $\rho \in C^{2+\delta}_0(\mathbb{R}^N_+)$, where $\delta \notin \mathbb{N}$ and $\delta > 0$. In addition $v \in C^{2+\delta}(\mathbb{R}^N_+)$ and is the unique solution of **(10)** satisfying

$$|v(x)| \leq C \frac{x_N}{1 + |x|^N} \quad \text{for } x \in \mathbb{R}^N,$$

and for some $C > 0$. This result is even more general since it is still true for the operator $(-\Delta)^s$, $s > 0$, when $2s + \delta \notin \mathbb{N}$ and instead of $G_{\mathbb{R}^N_+}$ is considered the Green's function for $(-\Delta)^s$ in $\mathbb{R}^N_+$.

We also refer to the work made by Bachar and Habib [8], in which using some inequalities for the Green's function $G_{\mathbb{R}^N_+}$, the class of potentials $K(\mathbb{R}^N_+)$, and the subclass $K^\infty(\mathbb{R}^N_+)$ which properly contains the classical Kato class $K^\infty_N(\mathbb{R}^N_+)$, they have showed that Problem **(9)** has a unique solution $u \in C^{2+\delta}(\mathbb{R}^N_+) \cap C_0(\mathbb{R}^N_+)$ when

$$\rho(x) \leq \frac{1}{(1+|x|)^{\mu-\gamma} x_N^\gamma} \quad \text{for } x \in \mathbb{R}^N_+,$$

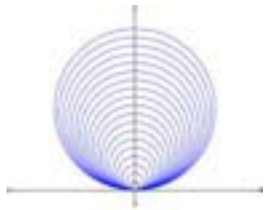
with $N+2 \leq \mu+\gamma$ and $\gamma < 2$, in addition

$$\frac{1}{C}\frac{x_N}{1+|x|^N} \leq u(x) \leq C\frac{x_N}{1+|x|^N} \quad \text{for } x \in \mathbb{R}^N,$$

if $\gamma < 1$, for some $C > 0$ (see [8, Theorem 4] and [8, Example 3]).

In our work, it should be noted that the technique used in the article by Brezis and Kamin cannot be used directly. However, we can use the following fact:

$$\mathbb{R}^N_+ = \bigcup_{n=1}^{\infty} B_n(a_n),$$



where, for each $n \in \mathbb{N}$, we have denoted by

$$B_n(a_n) := \{x \in \mathbb{R}^N : |x - a_n| < n\},$$

and $a_n := (0, \ldots, 0, n) \in \mathbb{R}^N_+$. This fact allows us give sufficient and necessary conditions to obtain existence of a bounded solution of Problem **(7)** by using a monotonicity argument involving Green's functions in the balls $B_n(a_n)$ and the Green's function in the half space. So, we begin by giving the following definition.

Definition. *Let $\rho \in L^\infty_{loc}\left(\mathbb{R}^N_+\right)$, $\rho(x) \geq 0$ and ρ not identically zero. We say that ρ has the property* (H$_+$) *if there exist a bounded solution of:*

$$-\Delta u = \rho(x) \text{ in } \mathbb{R}^N_+. \qquad \textbf{(P}_+\textbf{)}$$

In this way, here our main result is the next:

Theorem 6. *Let $\rho \in L^\infty_{loc}\left(\mathbb{R}^N_+\right)$, $\rho(x) \geq 0$ and ρ not identically zero. Then ρ satisfies property* (H$_+$) *iff*

$$\int_{\mathbb{R}^N_+} G_{\mathbb{R}^N_+}(x,y)\rho(y)dy \in L^\infty(\mathbb{R}^N_+).$$

To finish, we will give some applications of above theorem, which shows that Problem **(P$_+$)** has bounded positive solution for some ρ and also the nonexistence of bounded solution. For instance, if we consider

$$\rho(x) \leq \frac{1}{(1+|x|)^{\beta} x_N^\gamma} \quad \text{for } x \in \mathbb{R}^N_+,$$

with $0 \leq \gamma < 1$ and $2 < \beta + \gamma$. Then we will show that Problem **(9)** has a solution $u \in H^1(\mathbb{R}^N_+) \cap L^\infty(\mathbb{R}^N_+)$. Furthermore, if we impose $\beta + \gamma < N + 1$, we also show that the solution vanishes at infinity, that is to say

$$\lim_{|x|\to\infty} u(x) = 0 \quad \text{and} \quad \lim_{x_N \to 0} u(x) = 0.$$

Chapter 1

Basic results

1.1 Function spaces

We begin by giving some function spaces that will be presented throughout this work.

Definition 1.1.1. Let $\Omega \subset \mathbb{R}^N$ open subset of $\mathbb{R}^N$.

- $C^k(\Omega)$, $k = 1, 2, \dots$ is the space of functions $u : \Omega \to \mathbb{R}$ that are k times differentiable in Ω and whose k-th derivates are continuous in Ω.

- $C^\infty(\Omega)$, is the space of functions $u : \Omega \to \mathbb{R}$ that are infinitely many times differentiable in Ω.

- $C_0^\infty(\Omega)$, is the subspace of $C^\infty(\Omega)$ consisting of functions with compact support in Ω, where the support of a (continuous) function $u : \Omega \to \mathbb{R}$ is the closure (in $\mathbb{R}^N$) of the set $\{x \in \Omega : u(x) \neq 0\}$. Likewise, $C_0^k(\Omega)$ is the subset of $C^k(\Omega)$ containing only functions with compact support.

- $L^p(\Omega)$, for $p \in [1, \infty)$ is the Lebesgue space of measurable functions (Lebesgue measure) $u : \Omega \to \mathbb{R}$ such that $\int_\Omega |u(x)|^p < \infty$, while $L^\infty(\Omega)$ is the space of measurable functions such that $\operatorname{ess\,sup}_{x \in \Omega} |u(x)| < \infty$, where

$$\operatorname*{ess\,sup}_{x \in \Omega} |u(x)| = \inf\{C > 0 : |u(x)| \leq C \text{ a.e. in } \Omega\}.$$

  The norms that make $L^p(\Omega)$ Banach spaces are, respectively,

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)| dx \right)^{\frac{1}{p}} \quad \text{and} \quad \|u\|_\infty = \operatorname{ess\,sup} |u(x)|.$$

- $L^p_{loc}(\Omega)$, for $p \in [1, \infty)$ is the space of measurable functions $u : \Omega \to \mathbb{R}$ such that for every compact set $k \subset \Omega$ $\|u\|_{L^p(K)} < \infty$.

- $H^1(\Omega)$ is the Sobolev space defined by

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) : \frac{\partial u}{\partial x_i} \in L^2(\Omega), i = 1, \dots, N \right\},$$

where the derivate

$$D_i u = u_{x_i} = \frac{\partial u}{\partial x_i}, \text{ for } i = 1, \dots, N$$

is in the sense of distributions. It is a Hilbert space, when endowed with the scalar product given by

$$\langle u, v \rangle_{H^1(\Omega)} = \int_\Omega \big(\nabla u \nabla v + uv\big) dx.$$

Therefore, the corresponding norm is

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega \big(|\nabla u|^2 + u^2\big) dx \right)^{\frac{1}{2}}.$$

- $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$.
- $D^{1,2}(\mathbb{R}^N)$, for $N \geq 3$, is the space defined as follows:

$$D^{1,2}(\mathbb{R}^N) = \left\{ u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N) : \frac{\partial u}{\partial x_i} \in L^2(\Omega), i = 1, \dots, N \right\}.$$

This space has a Hilbert structure when endowed with the scalar product

$$\langle u, v \rangle_{D^{1,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \nabla u \nabla v dx.$$

So that the corresponding norm is

$$\|u\|_{D^{1,2}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{1}{2}}.$$

Remark 1.1.1. The space $C_0^\infty(\mathbb{R}^N)$ is dense in $D^{1,2}(\mathbb{R}^N)$. Moreover $H^1(\mathbb{R}^N) \subset D^{1,2}(\mathbb{R}^N)$, but there are functions, such as

$$u(x) = \frac{1}{(1 + |x|)^{\frac{N}{2}}}$$

that are in $D^{1,2}(\mathbb{R}^N)$ but not in $L^2(\mathbb{R}^N)$, and hence, not in $H^1(\mathbb{R}^N)$.

Definition 1.1.2. Let $V : \mathbb{R}^N \to \mathbb{R}$ a nonnegative function.

i) The Hilbert space $H_V^1\big(\mathbb{R}^N\big)$ is defined by

$$H_V^1\big(\mathbb{R}^N\big) = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x) u^2 dx < +\infty \right\}$$

with scalar product and norm given by

$$\langle u, v \rangle_{H_V^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \big(\nabla u \nabla v + V(x) uv\big) dx \quad \text{and} \quad \|u\|_{H_V^1(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \big(|\nabla u|^2 + V(x) u^2\big) dx \right)^{\frac{1}{2}}.$$

ii) We will denote by $E = H^1_V(\mathbb{R}^N) \times H^1_V(\mathbb{R}^N)$ the Hilbert space with the inner product given by

$$\langle (u,v),(\varphi,\phi)\rangle = \int_{\mathbb{R}^N} \Big(\nabla u \nabla \varphi + \nabla v \nabla \phi + V(x)u\varphi + V(x)v\phi\Big)dx$$

and corresponding norm

$$\|(u,v)\| = \left(\int_{\mathbb{R}^N} \Big(|\nabla u|^2 + v(x)u^2 + |\nabla v|^2 + V(x)v^2\Big)dx\right)^{\frac{1}{2}}.$$

Definition 1.1.3. Let $\Omega \subset \mathbb{R}^N$ open set, $\alpha \in (0,1)$ and $u : \Omega \to \mathbb{R}$. We said that u is **Hölder continous with exponent** α **in** Ω if there exist $C > 0$ such that

$$|u(x) - u(y)| \leq C|x-y|^\alpha, \quad x, y \in \Omega.$$

Definition 1.1.4. Likewise, we said that u is **locally Hölder continuous with exponent** α **in** Ω if u is Hölder continuous with exponent α on campact subset of Ω.

Definition 1.1.5. Let $\Omega \subset \mathbb{R}^N$ open set and $\alpha \in (0,1)$.

i) If $u : \Omega \to \mathbb{R}$ is bounded and continuous, we write

$$||u||_{C(\overline{\Omega})} := \sup_{x\in\Omega} |u(x)|.$$

ii) The α^{th}-Hölder seminorm of $u : \Omega \to \mathbb{R}$ is

$$[u]_{C^{0,\alpha}(\overline{\Omega})} := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left\{\frac{|u(x)-u(y)|}{|x-y|^\alpha}\right\}.$$

iii) The α^{th}-Hölder norm is

$$||u||_{C^{0,\alpha}(\overline{\Omega})} := ||u||_{C(\overline{\Omega})} + [u]_{C^{0,\alpha}(\overline{\Omega})}.$$

Definition 1.1.6. Let $\Omega \subset \mathbb{R}^N$ open set, $\alpha \in (0,1)$ and k a nonnegative integer. The **Hölder Space** $C^{k,\alpha}(\overline{\Omega})$ consisting of all functions $u \in C^k(\overline{\Omega})$ for which the norm

$$||u||_{C^{k,\alpha}(\overline{\Omega})} := \sum_{|\alpha|\leq k} ||Du||_{C(\overline{\Omega})} + \sum_{|\alpha|=k} [Du]_{C^{0,\alpha}(\overline{\Omega})}$$

is finite.

Remark 1.1.2.

i) The space $C^{k,\alpha}(\overline{\Omega})$ consists of those functions u that are k-times continuously differentiable and whose k^{th}-partial derivates are Hölder continuous with exponent α in Ω.

ii) The Hölder Space $C^{k,\alpha}(\Omega)$ are defined as the subspaces of $C^k(\Omega)$ consists of functions whose k^{th}- order partial derivates are locally Hölder continuous with exponent α in Ω.

iii) For Simplicity we write

$$C^{0,\alpha}(\overline{\Omega}) = C^{\alpha}(\overline{\Omega}), \quad C^{0,\alpha}(\Omega) = C^{\alpha}(\Omega).$$

vi) If $\alpha = 1$, $C^{\alpha}(\overline{\Omega})$ is often called the space of uniformly Lipschitz continuous functions. If $\alpha = 0$, $C^{k,0}(\overline{\Omega})$ (respectively $C^{k,0}(\Omega)$) are the usual $C^k(\overline{\Omega})$ (respectively $C^k(\Omega)$) spaces. Moreover, for $\alpha \in [0,1]$, $C_0^{k,\alpha}(\Omega)$ denotes the space of functions in $C^{k,\alpha}(\Omega)$ having compact support in Ω.

Theorem 1.1.7. *Let $\Omega \subset \mathbb{R}^N$ open set, $\alpha \in (0,1)$ and k a nonnegative integer. The space of functions $C^{k,\alpha}(\overline{\Omega})$ is a Banach space.*

1.2 Embeddings

We recall that a Banach space X is embedded continuously in a Banach space Y, which we denote by $X \hookrightarrow Y$, if

1. $X \subseteq Y$.

2. The canonical injection $i : X \to Y$ is a continuous (linear) operator. This means that there exists a constant $C > 0$ such that $\|i(u)\|_X \leq C\|u\|_X$.

A Banach space X is embedded compactly in a Banach space Y if X is embedded continuously in Y and the canonical injection i is a compact operator.
The followings results are the cases of the Sobolev and Rellich Embeddings theorems that we need is this work. First, we deal with functions defined on bounded sets.

Theorem 1.2.1. *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded subset of $\mathbb{R}^N$, with $N \geq 3$. Then*

$$H_0^1(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for every} \quad q \in \left[1, \frac{2N}{N-2}\right].$$

The embedding is compact if and only if $q \in \left[1, \frac{2N}{N-2}\right)$.

The number $\frac{2N}{N-2}$ is denoted by 2^* and is called the critical Sobolev exponent for the embedding of H_0^1 into L^q. The term critical refers to the fact that the embedding of the preceding theorem fails for $q > 2^*$.
For functions defined on general, unbounded domains, in view of our applications we limit ourselves to the case $\Omega = \mathbb{R}^N$.

Theorem 1.2.2. *Let $n \geq 3$. Then*

- $H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ *for every $q \in [2, 2^*]$.*

- $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$.

These embeddings are never compact.

Remark 1.2.1. We point out that the continuity of above embeddings is expressed explicitly by inequalities if the form

$$\|u\|_{L^q(\mathbb{R}^N)} \leq C\|u\|_{H^1(\mathbb{R}^N)}$$

where C does not depend on u.

Below, we will give conditions presented in [3] to get a embedding result (see [3, Theorem 5])

Proposition 1.2.3. *Let $N \geq 3$. Assume the following hypothesis on V and ρ*

- *$V : \mathbb{R}^N \to \mathbb{R}$ is smooth, there exists $a, A > 0$, $\alpha \in (0, 2]$ such that*

$$\frac{a}{1+|x|^\alpha} \leq V(x) \leq A, \quad \text{for all } x \in \mathbb{R}^N;$$

- *$\rho : \mathbb{R}^N \to \mathbb{R}$ is smooth and there exist $k > 0$, $\beta \geq \alpha$, such that*

$$0 < \rho(x) \leq \frac{k}{1+|x|^\beta}, \quad \text{for all } x \in \mathbb{R}^N.$$

Then, the embedding

$$H^1_V(\mathbb{R}^N) \hookrightarrow L^q_\rho(\mathbb{R}^N)$$

is continuous for $2 \leq q \leq 2^*$ and compact if $2 < q < 2^*$, where we denote by $L^q_\rho(\mathbb{R}^N)$, $q > 1$, the weighted Lebesgue space

$$L^q_\rho(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} : \quad u \text{ is measurable and } ||u||_{L^q_\rho(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} \rho(x)|u|^q dx \right)^{\frac{1}{q}} < +\infty \right\}.$$

1.3 Frequently used results

Now, we will give some classic results that we will use throughout this work.

Theorem 1.3.1. (Green's identity) *Let $\Omega \subset \mathbb{R}^N$ be open, bounded and smooth.*

i) (Gauss-Green theorem). *Let $v \in C^1(\overline{\Omega})$, then*

$$\int_\Omega u_{x_i} dx = \int_{\partial\Omega} \nu^i dS \quad \text{for } i = 1, \ldots, N.$$

ii) (Integration by parts formula). *Let $u, v \in C^1(\overline{\Omega})$, then*

$$\int_\Omega u_{x_i} v dx = -\int_\Omega u v_{x_i} dx + \int_{\partial\Omega} u v \nu^i dS \quad \text{for } i = 1, \ldots, N.$$

Green's formulas:

iii) Let $u \in C^2(\overline{\Omega})$, then

$$\int_\Omega \Delta u dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS.$$

iv) Let $u \in C^2(\overline{\Omega})$ and $v \in C^1(\overline{\Omega})$. Then

$$\int_\Omega v\Delta u dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v dS - \int_\Omega \nabla u \cdot \nabla v \; dx.$$

v) Let $u, v \in C^2(\overline{\Omega})$, then

$$\int_\Omega (u\Delta v - v\Delta u)\, dx = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) dS,$$

where along $\partial\Omega$ is defined the outward pointing unit normal vector field $(\nu^1, \ldots, \nu^N)$, $\nu = \nu(x)$ is the outward normal to $\partial\Omega$ at x, $\frac{\partial u}{\partial \nu}(x) = \nabla u(x) \cdot \nu(x)$ and where S is the surface measure on $\partial\Omega$.

Theorem 1.3.2. (Monotone convergence) *Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ be a measurable set Lebesgue, and let (u_n) be a sequence increase of measurable nonnegative functions such that for each $x \in \Omega$ there exists $\lim_{n\to\infty} u_n(x)$. Then*

$$\int_\Omega \lim_{n\to\infty} u_n dx = \lim_{n\to\infty} \int_\Omega u_n dx.$$

Lemma 1.3.3. (Fatou's lemma) *Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ a measurable set Lebesgue, and let (u_n) be a sequence of measurable nonnegative functions such that for each $x \in \Omega$ there exists $\liminf_{n\to\infty} u_n(x)$. Then*

$$\int_\Omega \liminf_{n\to\infty} u_n dx \leq \liminf_{n\to\infty} \int_\Omega u_n dx.$$

Theorem 1.3.4. (Lebesgue's dominated convergence) *Let $\Omega \subset \mathbb{R}^N$ be open and let $(u_n) \subset L^1(\Omega)$ be a sequence such that*

1. *$u_n(x) \to u(x)$ a.e. in Ω as $n \to \infty$.*
2. *There exists $v \in L^1(\Omega)$ such that for all n, $|u_n(x)| \leq v(x)$ a.e. in Ω.*

Then $u \in L^1(\Omega)$ and

$$\int_\Omega u dx = \lim_{n\to\infty} \int_\Omega u_n dx.$$

Theorem 1.3.5. *Let $\Omega \subset \mathbb{R}^N$ be open and let $(u_n) \subset L^p(\Omega)$, $p \in [1, \infty]$, be a sequence such that $u_n \to u$ in $L^p(\Omega)$ as $n \to \infty$. Then there exists a subsequence (u_{n_k}) and a function $v \in L^p(\Omega)$ such that*

1. *$u_{n_k}(x) \to u(x)$ a.e. in Ω as $n \to \infty$.*
2. *For all k, $|u_{n_k}(x)| \leq v(x)$ a.e. in Ω.*

Theorem 1.3.6. (Fubini) *Let Ω_1 and Ω_2 be σ-finite measure spaces and suppose $u(x, y)$ is $\Omega_1 \times \Omega_2$ measurable. If either*

$$\int_{\Omega_1} \left(\int_{\Omega_2} |u(x,y)| dy \right) dx < \infty \quad \text{or} \quad \int_{\Omega_2} \left(\int_{\Omega_1} |u(x,y)| dx \right) dy < \infty$$

then

$$\int_{\Omega_1\times\Omega_2} |u(x,y)|dxdy < \infty$$

and

$$\int_{\Omega_1}\left(\int_{\Omega_2}|u(x,y)|dy\right)dx = \int_{\Omega_1\times\Omega_2}|u(x,y)|dxdy = \int_{\Omega_2}\left(\int_{\Omega_1}|u(x,y)|dx\right)dy.$$

Theorem 1.3.7. (Poincaré inequality) *Let $\Omega \subset \mathbb{R}^N$ be open and bounded. Then there exists a constant $C > 0$, depending only on Ω, such that*

$$\int_\Omega u^2 dx \le C\int_\Omega |\nabla u|^2 dx \quad \textit{for all} \quad u \in H_0^1(\Omega).$$

Therefore, the quantity $\left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}$ is a norm on $H_0^1(\Omega)$, equivalent to the standard one.

Theorem 1.3.8. (Banach-Alaoglu) *Let X be a reflexive Banach space. If $B \subset X$ is bounded, then B is relatively compact in the weak topology of X.*

In a Banach space X with topological dual X', we write $u_n \to u$ when the sequence (u_n) converges strongly to u, that is, in the strong topology of X, which means that $\|u_n - u\|_X \to 0$ as $n \to \infty$; we write $u_n \rightharpoonup u$ if u_n converges weakly to u, i.e. in the weak topology of X, which means that

$$f(u_n) \to f(u) \quad \text{as} \quad n \to \infty \quad \text{for all} \quad f \in X'.$$

Example 1.3.9. The following chain of arguments is used very frequently, often automatically. Let $\Omega \subset \mathbb{R}^N$ be open and bounded. Suppose that a sequence $(u_n) \subset H_0^1(\Omega)$ satisfies

$$\int_\Omega |\nabla u_n|^2 dx \le C \quad \text{for all} \quad n \in \mathbb{N}$$

and for some $C > 0$ independedt of k. By **Theorem** 1.3.7, the sequence (u_n) is bounded in $H_0^1(\Omega)$. The space $H_0^1(\Omega)$, being a Hilbert space, is reflexive. Therefore, by Banach-Alaoglu the Theorem sequence is relatively compact in $H_0^1(\Omega)$ endowed with the weak topology. This means that there exists $u \in H_0^1(\Omega)$ and a subsequence that, again, call u_n, such that

$$u_n \rightharpoonup u \quad \text{in} \quad H_0^1(\Omega).$$

By **Theorem** 1.2.1, the embedding of $H_0^1(\Omega)$ into $L^q(\Omega)$ is compact for every $q \in \left[1, \frac{2N}{N-2}\right)$. Then we can say that

$$u_n \to u \quad \text{in} \quad L^q(\Omega) \quad \text{for every} \quad q \in \left[1, \frac{2N}{N-2}\right).$$

By **Theorem** 1.3.5 there exists another subsequence, still denoted u_n, and there exists $v \in L^q(\Omega)$, such that

1. $u_n(x) \to u(x)$ a.e. in Ω
2. $|u_n(x)| \le v(x)$ a.e. in Ω for all $n \in \mathbb{N}$.

Remark 1.3.1. This same argument can be used in the whole space $\mathbb{R}^N$ since the spaces in which we will work later will be given, and for them, there is an embedding result equivalent to **Theorem 1.2.1**.

Next, we give a result which converts N - dimensional integrals into integrals over spheres.

Theorem 1.3.10. (Polar coordinates)

i) Let $u : \mathbb{R}^N \to \mathbb{R}$ *be continuous and summable. Then*

$$\int_{\mathbb{R}^N} u(x)dx = \int_0^\infty \left(\int_{\partial B(x_0,r)} u(x)dS(x) \right) dr,$$

for each point $x_0 \in \mathbb{R}^N$

ii) In particular

$$\frac{d}{dr}\left(\int_{B(x_0,r)} u(x)dx \right) = \int_{\partial B(x_0,r)} u(x)dS(x),$$

for each $r > 0$.

Remark 1.3.2. Let $R > 0$. If there exists a function $g : (0, R) \to \mathbb{R}$ such that $u(x) = g(|x|)$, from $ii)$ follows that

$$\begin{aligned} \int_{B(0,R)} u(x)dx &= \int_0^R \left(\int_{\partial B(0,r)} u(x)dS(x) \right) dr \\ &= \int_0^R g(r) \left(\int_{\partial B(0,r)} dS(x) \right) dr \\ &= N w_N \int_0^R g(r) r^{N-1} dr, \end{aligned}$$

where

$$w_N = \int_{\partial B(0,1)} dS(x).$$

Thus, we have

$$\int_{B(0,R)} u(x)dx = N w_N \int_0^R g(r) r^{N-1} dr.$$

1.4 Differential calculus in Banach spaces

We present a short review of the main definitions and results concerning the differential calculus for real functionals defined in Banach space. A complete discussion of this topic and more generalities of differential calculus in normed spaces can be found in [2], and you can also see [41].

Definition 1.4.1. Let X be a Banach space, $U \subseteq X$ an open set and let $I : U \to \mathbb{R}$ be a functional

- We say that I is Gâteaux differentiable at $u \in U$ if there exists $A \in X'$ such that, for all $v \in X$,
$$\lim_{t \to 0} \frac{I(u + tv) - I(u)}{h} = Av. \tag{1.4.1}$$

- We also say that I is Fréchet differentiable at $u \in U$ if there exists $A \in X'$ such that
$$\lim_{\|v\| \to 0} \frac{I(u + v) - I(u) - Av}{\|v\|} = 0. \tag{1.4.2}$$

Remark 1.4.1.

1. If I is Gâteaux differentiable at u, there is only one linear functional $A \in X'$ satisfying **(1.4.1)**. It is called the Gâteaux differential of I at u and is denoted by $I'_G(u)$.

2. If the functional is Gâteaux differentiable at every $u \in U$, we say that I is Gâteaux differentiable on U.

3. The map $I'_G : U \to X'$ that sends $u \in U$ to $I'_G(u) \in X'$ is called the Gâteaux derivate of I.

4. If I is Fréchet differentiable at u, the unique element of X'such that **(1.4.2)** holds is called the (Fréchet) differential of I at u and is denoted by $I'(u)$.

5. If the functional I is differentiable at every $u \in U$, we say that I is differentiable on U.

6. The map $I' : U \to X'$ that sends $u \in U$ to $I'(u) \in X'$ is called the (Fréchet) derivate of I.

7. If the derivate I' is continuous from U to X' we say that I is of class C^1 on U and we write $I \in C(U)$.

Definition 1.4.2. Let $(H, \langle , \rangle)$ be a Hilbert space, $U \subseteq H$ an open set, and let $R : H \to H$ be the Riesz isomorphism. Assume that the functional $I : \to \mathbb{R}$ is differentiable at u. The element $RI'(u) \in H$ is called the gradient of I at u and is denoted by $\nabla I(u)$; therefore

$$I'(u)v = \langle \nabla I(u), v \rangle \ \text{ for every } \ v \in H.$$

We have the following classic result (see [2]).

Lemma 1.4.3. *Assume that $U \subseteq X$ is an open set, that I is Gâteaux differentiable on U and I'_G is continuous at $u \in U$. Then I is also differentiable at u, and $I'_G(u) = I'(u)$.*

We conclude by giving the definitions of critical points and critical levels that will be one of the main themes in studied in this work.

Definition 1.4.4. Let X a Banach space, $U \subseteq X$ is an open set and assume that $I : U \to \mathbb{R}$ is differentiable.

- A critical point of I is a point $u \in U$ such that $I'(u) = 0$.

- If $I'(u) = 0$ at $I(u) = c$, we say that u is a critical point for I at level c and c is a critical value of I.

- If for some $c \in \mathbb{R}$ the set $I^{-1}(c) \subset X$ contains at least a critical point, we sat that c is a critical level for I.

Remark 1.4.2.

1. As $I'(u)$ is an element of the dual space X', u is a critical point of I if $I'(u)v = 0$ for all $v \in X$.

2. The equation $I'(u) = 0$ is called the Euler-Lagrange equation associated to the functional I.

Next, we will give an example which will be used automatically in this work. For this purpose, we will give the definition of the Carathéodory condition and a Nemytskii operator.

Definition 1.4.5. Let $\Omega \subset \mathbb{R}^N$. We say that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function if satisfies

- For all $t \in \mathbb{R}$ the function $x \mapsto f(x,t)$ is measurable.

- For almost all $x \in \Omega$ the function $t \mapsto f(x,t)$ is continuous.

Remark 1.4.3. It is clear that if $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, then $x \mapsto g(x, u(x))$ is measurable for every measurable $u : \Omega \to \mathbb{R}$.

Proposition 1.4.6. *Let $\Omega \subset \mathbb{R}^N$ be open. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ a Carathéodory function such that for some $p, q \geq 1$, $c > 0$ and $a \in L^q(\Omega)$*

$$|f(x,t)| \leq a(x) + c|t|^{\frac{p}{q}}.$$

Then, the Nemytskii operator $g : L^p(\Omega) \to L^q(\Omega)$ defined by

$$f(u)x = f(x, u(x))$$

is continuous.

Proof. Since

$$|f(x,u(x))|^q \leq \left|a(x) + c|u(x)|^{\frac{p}{q}}\right|^q \leq 2^{q-1}\Big(|a(x)|^q + c|u(x)^p|\Big) \in L^1(\Omega),$$

we have $f(x, u(x)) \in L^q(\Omega)$. Let $u_k \to u$ in $L^p(\Omega)$. Now, we will show that $g(u_k) \to g(u)$ in $L^q(\Omega)$. In fact, let (u_n) a subsequence of (u_k). Then, from **Theorem 1.3.5** there exists a subsequence, still denoted u_n, and there exists $v \in L^q(\Omega)$, such that

1. $u_n(x) \to u(x)$ a.e. in Ω

2. $|u_n(x)| \leq v(x)$ a.e. in Ω for all $n \in \mathbb{N}$.

Thus $f(x, u_n(x)) \to f(x, u(x))$ a.e. in Ω. Since

$$|f(x,u_n(x))|^q \leq 2^{q-1}\Big(|a(x)|^q + c|v(x)^p|\Big) \in L^1(\Omega),$$

we see that $|f(x,u_n(x)) - f(x,u(x))| \in L^q(\Omega)$. Therefore, by dominated convergence we have $g(u_n) \to g(u)$ in $L^q(\Omega)$ and thus $g(u_k) \to g(u)$ in $L^q(\Omega)$. □

Example 1.4.7. Let $\Omega \subset \mathbb{R}^N$ be open. Suppose p and q are conjugate exponents, $1 < p < \infty$. Assume $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, there exist $c > 0$ and $a \in L^q(\Omega)$ such that

$$|f(x,t)| \leq a(x) + c|t|^{\frac{p}{q}}.$$

Define $F : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$F(x,t) = \int_0^t f(x,t)dt.$$

Then, the functional $\psi : L^p(\Omega) \to \mathbb{R}$ defined as

$$\psi(u) = \int_\Omega F(x,u(x))dx$$

is of class $C^1(L^p(\Omega), \mathbb{R})$ and

$$I'(u)v = \int_\Omega f(x,u(x))v(x)dx.$$

Proof. Let $u, v \in L^p(\Omega)$, $x \in \Omega$ and $t \in [0,1]$. By the mean value theorem there is $\xi \in (0,1)$ such that

$$F(x,u(x)+tv(x)) - F(x,u(x)) = f(x,u(x)+\xi v(x))v(x).$$

Since

$$\begin{aligned} |f(x,u(x)+\xi v(x))|^q &\leq \Big(a(x) + c|u(x)+v(x)|^{\frac{p}{q}}\Big)^q \\ &\leq 2^{q-1}\Big(a(x)^q + |u(x)^p| + c|v(x)^p|\Big) \in L^1(\Omega), \end{aligned}$$

by Holder inequality, we have $f(x,u(x)+\xi v(x))v(x) \in L^1(\Omega)$. Therefore from dominated convergence we see that

$$\lim_{t\to 0}\int_\Omega \frac{F(x,u(x)+tv(x)) - F(x,u(x))}{t}dx = \int_\Omega f(x,u(x))v(x)dx.$$

This shows that ψ is Gâteaux differentiable. Moreover, **Proposition** **1.4.6** say that the map $u \mapsto f(x,u(x))$ is continuous from $L^p(\Omega)$ to $L^q(\Omega)$, then by **Lemma** **1.4.3** we conclude that ψ is differentiable. □

1.5 Lower and upper solutions

In this section the classic result of existence of solution of the problem will be given:

$$\begin{cases} -\Delta u = f(x,u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.5.1}$$

between an upper and a lower solution, where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$, $u : \overline{\Omega} \to \mathbb{R}$ and $f : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ are given functions.

We begin by introducing the definition of an upper solution and a lower solution of Problem **(1.5.1)**.

Definition 1.5.1. Let Ω a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$.

i) A function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is called **classic solution** (**upper solution**, **lower solution**) of Problem **(1.5.1)**, if:

$$\begin{cases} -\boldsymbol{\Delta u} = (\geq, \leq)\, \boldsymbol{f(x,u)} & \text{in} \quad \boldsymbol{\Omega} \\ \boldsymbol{u} = (\geq, \leq)\, \boldsymbol{0} & \text{on} \quad \boldsymbol{\partial\Omega}. \end{cases}$$

ii) A function $u \in H^1(\Omega)$ is called **weak solution** (**weak upper solution**, **weak lower solution**) of Problem **(1.5.1)**, if:

$$\begin{cases} \int_{\Omega} \boldsymbol{\nabla u \nabla \varphi dx} = (\geq, \leq) \int_{\Omega} \boldsymbol{f(x,u)\varphi dx} & \text{in} \quad \boldsymbol{\Omega} \\ \boldsymbol{u} = (\geq, \leq)\, \boldsymbol{0} & \text{on} \quad \boldsymbol{\partial\Omega}, \end{cases}$$

for all $\varphi \in C_0^\infty(\Omega)$.

Remark 1.5.1. If $u \in C^2(\Omega) \cap (\overline{\Omega})$ is a weak upper solution (respectively, weak lower solution) of **(1.5.1)**, then

$$-\Delta u \geq f(x,u) \text{ (resp.} -\Delta u \leq f(x,u)) \text{ in } \Omega,$$

that is, u is a usual upper solution (resp. lower solution) of **(1.5.1)**.

The following result gives (see [30]) us the existence of solution of Problem **(1.5.1)**, proving that there is an upper and a lower solution. More precisely:

Theorem 1.5.2. *Suppose that $f \in C(\overline{\Omega} \times \mathbb{R})$ and there exist $\underline{u}, \overline{u} \in H^1(\Omega) \cap C(\overline{\Omega})$ weak lower and weak upper solutions (resp.) of Problem* ***(1.5.1)*** *such that $\underline{u}(x) \leq \overline{u}(x)$ for all $x \in \Omega$. Then, the Problem* ***(1.5.1)*** *has at least a weak solution $u \in H_0^1(\Omega)$ such that*

$$\underline{u}(x) \leq u(x) \leq \overline{u}(x) \text{ in } \Omega.$$

Since here we are working with weak solutions, it is necessary to give the following result, which is a weak maximum principle.

Lemma 1.5.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $m \geq 0$ be a constant. If u is a continuous function in $\overline{\Omega}$ which is nonnegative on $\partial\Omega$ and satisfies*

$$\int_{\Omega} (u\Delta\varphi - mu\varphi)dx \leq 0, \tag{1.5.2}$$

for all $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$. Then $u \geq 0$ in $\overline{\Omega}$.

Proof. Suppose by contradiction that $\min_{x\in\overline{\Omega}} u(x) < 0$. There exists an $x_0 \in \Omega$ such that

$$u(x_0) = \min_{x\in\overline{\Omega}} u(x).$$

As u is continuous, there is $B(x_0, r) \subset \Omega$ such that $u(x) < 0$ in $B(x_0, r)$. Let $B = B(x_0, r/2)$. Choose a nonnegative $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} \varphi dx = 1 \text{ and put } \varphi_\varepsilon(x) = \frac{1}{\varepsilon^N}\varphi\left(\frac{x}{\varepsilon}\right) \text{ for } \varepsilon > 0.$$

Define

$$u_\varepsilon(x) = \int_\Omega u(y)\varphi_\varepsilon(x-y)dy, \quad x \in B \text{ for } \varepsilon > 0.$$

Then $u_\varepsilon \to u$ uniformly on B as $\varepsilon \to 0$, and so, for any $\eta > 0$ there is an $\varepsilon_0 > 0$ such that $u_\varepsilon - u \geq -\eta$ on ∂B for $0 < \varepsilon < \varepsilon_0$. Take a harmonic function h in B satisfying $h = u$ on ∂B, and using **(1.5.2)** we have

$$\Delta(u_\varepsilon(x) - h(x)) = \Delta u_\varepsilon(x) = \int_\Omega u(y)\Delta\varphi_\varepsilon(x-y)dy \leq m\int_\Omega u(y)\varphi_\varepsilon(x-y)dy \leq 0 \text{ in } B.$$

And for $\varepsilon > 0$ enough small $u_\varepsilon(x) - h(x) = 0$ on ∂B. Then, by the weak maximum principle we have $u_\varepsilon - h \geq -\eta$ in $\overline{B}$ and letting $\varepsilon \to 0$, $\eta \to 0$, we obtain $u \geq h$ in $\overline{B}$. From this inequality it follows that

$$u(x_0) \geq h(x_0) \geq \min_{x\in\partial B} h(x) = \min_{x\in\partial B} u(x) \geq \min_{x\in\overline{\Omega}} = u(x_0).$$

Therefore, $\min\limits_{x\in\partial B} h(x) = h(x_0)$. Then by the harmonicity of h and strong maximum principle we have

$$u \equiv u(x_0) \text{ on } \partial B.$$

Since $r > 0$ can be chosen arbitrarily small, we conclude that $u \equiv u(x_0)$ near x_0, and this implies that the set $M = \{x \in \Omega : u(x) = u(x_0)\}$ is open in Ω. Obviously, M is closed, so that we must have $M = \Omega$, that is, $u \equiv u(x_0)$ in Ω. This, however, contradicts the fact that $u \geq 0$ on $\partial\Omega$, and the proof is complete. □

1.6 Elliptic systems

In this section we present the type of systems of equations that we will work on in the following sections. Next we will give its properties and basic definitions. Consider the following system

$$\begin{cases} -\Delta u + V_1(x)u = f(x,u,v) & \text{in} \quad \Omega \\ -\Delta v + V_2(x)v = g(x,u,v) & \text{in} \quad \Omega \\ u = v = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.6.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$, $u, v, V_1, V_2 : \overline{\Omega} \to \mathbb{R}$ and $f, g : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ are given functions.

Definition 1.6.1. By a solution (upper solution, lower solution) of System **(1.6.1)** we mean a couple (u, v) in $(H^1(\Omega))^2$ satisfying

$$\begin{cases} -\Delta u + V_1(x)u = (\geq, \leq)\ f(x,u,v) & \text{in} \quad \Omega \\ -\Delta v + V_2(x)v = (\geq, \leq)\ g(x,u,v) & \text{in} \quad \Omega \\ u = (\geq, \leq)\ 0 & \text{on} \quad \partial\Omega \end{cases}$$

almost everywhere in x.
We say that (u, v) is nonnegative (positive) in Ω if each coordinate is.

The following general result due to Montenegro [35] establishes the existence of a solution for a systems like **(1.6.1)**, provided an upper solution and a lower solution exist.

Lemma 1.6.2. *[35, Lemma 2.1] Let $p_0 \geq N$ and $X := \{(u,v) \in C(\overline{\Omega})^2 : u = v = 0 \text{ on } \partial\Omega\}$ endowed with the norm $\|(u,v)\|_X = \|u\|_{C(\overline{\Omega})} + \|u\|_{C(\overline{\Omega})}$. Assume that there are a nonnegative lower solution $(\underline{u}, \underline{v})$ and a nonnegative upper solution $(\overline{u}, \overline{v})$ of the System* ***(1.6.1)*** *satisfying $\underline{u} \leq \overline{u}$ and $\underline{v} \leq \overline{v}$ in Ω. Set $M = \max\{\|(\underline{u}, \underline{v})\|_X, \|(\overline{u}, \overline{v})\|_X\}$. Let f and g be nonnegative Carathéodory functions such that $f(x, t_1, s_1) \leq f(x, t_2, s_2)$ and $g(x, t_1, s_1) \leq g(x, t_2, s_2)$ for every $x \in \Omega$, $0 \leq t_1 \leq t_2 \leq M$, $0 \leq s_1 \leq s_2 \leq M$ and $\sup\{f(\cdot, t, s) : \ t, s \in [0, M]\}$, $\sup\{g(\cdot, t, s) : \ t, s \in [0, M]\}$* are in *$L^{p_0}(\Omega)$. Then, System* ***(1.6.1)*** *admits a solution (u, v) verifying $\underline{u} \leq u \leq \overline{u}$ and $\underline{v} \leq v \leq \overline{v}$ in Ω.*

Next, we will give the definition of the types of variational systems that we will work on throughout this work.

Definition 1.6.3. The system

$$\begin{cases} -\boldsymbol{\Delta u} + \boldsymbol{V(x)u} = \boldsymbol{f(x,u,v)} & \text{in} \quad \mathbb{R}^{\boldsymbol{N}} \\ -\boldsymbol{\Delta v} + \boldsymbol{V(x)v} = \boldsymbol{g(x,u,v)} & \text{in} \quad \mathbb{R}^{\boldsymbol{N}} \end{cases} \tag{1.6.2}$$

is variational if either one of the following conditions hold:

a) There exists a differentiable function $G : \mathbb{R}^{N+2} \to \mathbb{R}$ such that

$$\frac{\partial G}{\partial u} = f \quad \text{and} \quad \frac{\partial G}{\partial v} = g.$$

In this case, the system is said to be *gradient*.

b) There exists a differentiable function $F : \mathbb{R}^{N+2} \to \mathbb{R}$ such that

$$\frac{\partial F}{\partial u} = g \quad \text{and} \quad \frac{\partial F}{\partial v} = f.$$

In this case, the system is said to be *Hamiltonian*.

The variational terminology comes from the fact that in both case, System **(1.6.2)** has a functional naturally associated with the system. In fact, if we work with functions $(u, v) \in H$, the functional associated with the gradient system is

$$I(u,v) = \frac{1}{2}\int_{\mathbb{R}^N} \Big(|\nabla u|^2 + V(x)u^2 + |\nabla v|^2 + V(x)v^2 \Big) dx - \int_{\mathbb{R}^N} G(x,u,v)dx,$$

while the one associated to a Hamiltonian system is

$$J(u,v) = \int_{\mathbb{R}^N} \Big(\nabla u \nabla v + V(x)uv \Big) dx - \int_{\mathbb{R}^N} F(x,u,v)dx.$$

These two types of variational systems can be treated using the critical point theory, since the critical points of their functionals are solutions of the System **(1.6.2)**. There are several methods to tackle this question. The most successful one in our framework seems to be the Mountain Pass Theorem of Ambrosetti and Rabinowitz (see [27] or [43, Theorem 1.17]) and the linking theorem ; here we follow [24, Theorem 2.1].

Definition 1.6.4. Let $(E, \|\cdot\|)$ be a real Hilbert space, $I \in C^1(E, \mathbb{R})$ and $c \in \mathbb{R}$. We call I satisfies a Palais-Smale condition level c if every sequence $(u_n) \subset E$ such that

$$I(u_n) \underset{n\to\infty}{\longrightarrow} c \quad \text{and} \quad I'(u_n) \underset{n\to\infty}{\longrightarrow} 0$$

has a strongly convergent subsequence in E.

Theorem 1.6.5. ***(Mountain Pass Theorem)***
Let $(E, \|\cdot\|)$ be a real Hilbert space, and $I : E \to \mathbb{R}$ a functional C^1 satisfying the Palais-Smale condition. Asume

M1) $I(0) = 0$.

M2) There exists a constants $r, a > 0$ such that

$$I(u) \geq a \text{ if } \|u\| = r.$$

M3) There exists $v \in E$ with

$$I(v) \leq 0 \text{ if } \|v\| > r.$$

Define

$$\Gamma := \{g \in C([0,1]; E) : g(0) = 0, g(1) = v\}.$$

Then $c = \inf_{g\in\Gamma} \max_{0\leq t\leq 1} I(g(t))$ is a critical value of I, greater than or equal to a.

Now, we give the linking result. For this purpose we begin by defining what is a cerami sequence.

Definition 1.6.6. Let $(E, \|\cdot\|)$ be a real Hilbert space, $J \in C^1(E, \mathbb{R})$ and $c \in \mathbb{R}$. We call a sequence $(u_n) \subset E$ a Cerami sequence at level c and denote $(C)_c$ for short, if

$$J(u_n) \underset{n\to\infty}{\longrightarrow} c \quad \text{and} \quad (1 + \|u_n\|)J'(u_n) \underset{n\to\infty}{\longrightarrow} 0$$

and we say that J satisfies the Cerami condition if every $(C)_c$ sequence has a strongly convergent subsequence in E.

Before proceeding, we recall some terminology introduced in [24, 29]. Let E^- be a closed subspace of a separable Hilbert space E with norm $\|\cdot\|_E$ and let $E^+ := (E^-)^\perp$. For $u \in E$ we shall write $u = u^+ + u^-$, where $u^\pm \in E^\pm$. On H we define a new norm

$$\|u\|_\tau := \max\left\{ \|u^+\|_E, \sum_{k=1}^{\infty} \frac{1}{2^k} |\langle u^-, e_k\rangle| \right\},$$

where $\{e_k\}$ is a total orthonormal sequence in E^-. The topology induced by $\|\cdot\|_\tau$ is called the τ-topology. We recall from [29] that a homotopy $h = I - g : A \times [0,1] \to E$ is called admissible, with $A \subset E$, if

i) h is τ-continuous, which means, $h(u_n, s_n) \to h(s, u)$ in τ-topology as $n \to \infty$ whenever $u_n \to u$ in τ-topology and $s_n \to s$ as $n \to \infty$;

ii) g is τ-locally finite-dimensional, i.e., for each $(u,s) \in A \times [0,1]$ there is a neighborhood U of (u,s) in the product topology of (E,τ) and $[0,1]$ such that $g(U \cap (A \times [0,1]))$ is contained in a finite dimensional subspace of E.

Notice that admissible homotopies are continuous in the strong topology. Also, if $\{u_m\}$ is a bounded sequence in E, then $u_m \to u$ in the τ-topology if, and only if, $u_m \rightharpoonup u$ in E^- and $u_m \to u$ in E^+.

The next proposition was proved in [24] and it is a generalization for $(C)_c$ sequences of [29, Theorem 3.4], in which a similar result was proved for Palais-Smale sequences.

Theorem 1.6.7. ***(Linking Theorem)***
Let $E = E^+ \oplus E^-$ be a separable real Hilbert space with E^- orthogonal to E^+ and $I \in C^1(E,\mathbb{R})$. Suppose

i) $J(z) = \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) - I(z)$, where $I \in C^1(E,\mathbb{R})$ is bounded from below, weakly sequentially lower semicontinuous and I' is weakly sequentially continuous.

ii) There exist $z_0 \in E^+ \setminus \{0\}$, $a > 0$ and $R > r > 0$ such that $J|_{N_r} \geq \alpha$ and $J|_{\partial M_{R,z_0}} \leq 0$.

Then, there exists a Cerami sequence for J at level $c := \inf_{h\in\Gamma} \sup_{u \in M_{R,z_0}} J(h(u,1))$ where

$$\Gamma := \Big\{ h \in C(M)\ ;\ h \text{ is admissible, } h(u,0) = u \text{ and } J(h(u,s)) \leq \max\{\Phi(u), -1\},\ \ \forall\, s \in [0,1] \Big\}$$

with

$$M_{R,z_0} = \{z = z^- + tz_0 :\ \|z\| \leq R,\ t \geq 0\}, \quad N_r = \{z \in E^+ :\ \|z\| = r\}, \quad M = M_{R,z_0} \times [0,1]$$

Moreover $c \geq a$.

Chapter 2

A scalar problem

2.1 The classic problem of Brezis-Kamin

This section will develop the main results obtained in the celebrated paper of Brezis and Kamin [9].
The objective is to give the necessary and sufficient conditions which guarantee the existence of a bounded positive solution of the problem

$$-\Delta u = \rho(x)u^{\alpha} \quad \text{in } \mathbb{R}^N, \quad N \geq 3 \tag{2.1.1}$$

where $0 < \alpha < 1$, $\rho \in L^{\infty}_{loc}(\mathbb{R}^N)$, $\rho(x) \geq 0$ and ρ not identically zero.

It is important noting that to obtain existence results of positive bounded solutions of Problem **(2.1.1)** is essential to impose the decay hypotheses on the weight $\rho(x)$. In fact, note that when $\rho(x) = 1$, Problem **(2.1.1)** is given by

$$-\Delta u = u^{\alpha} \quad \text{in } \mathbb{R}^N$$

and through the classic Liouville theorems, for $0 < \alpha < 2^* - 1$, Problem **(2.1.1)** has only a nonnegative C^2 solution given by $u \equiv 0$. This result was proved by Gidas-Spruck [24] in the case $1 < \alpha < 2^* - 1$. Moreover a proof using the method of moving parallel planes was given by Chen-Li [12], and it is valid in the whole range of α.
Therefore, the goal of this section is given the class of functions ρ so that Problem **(2.1.1)** has a bounded positive solution. For this purpose, we begin to talk about the problem

$$-\Delta u = \rho(x) \quad \text{in } \Omega$$

where Ω is a domain in $\mathbb{R}^N$. From classical theory, if $\rho \in C^2(\overline{\Omega})$, we know that the Newtonian potential of ρ, w defined on $\mathbb{R}^N$ by

$$w(x) = \int_{\Omega} \Gamma(x-y)\rho(y)dy,$$

where Γ is the fundamental solution of the Laplace equation, given by

$$\boldsymbol{\Gamma(x) := \begin{cases} -\dfrac{1}{2\pi} \ln |x| & \text{if} \quad N = 2 \\ \dfrac{1}{N(N-2)w_N} \dfrac{1}{|x|^{N-2}} & \text{if} \quad N \geq 3, \end{cases}}$$

defined for $x \in \mathbb{R}^N$, $x \neq 0$, belongs to $C^2(\Omega)$ and satisfies $-\Delta w = \rho$ in Ω (see [30]). However, if f is merely continuous, then w is not necessarily twice differentiable. In fact, take $0 < R < 1$ and $\Omega = B_R$, where we have denoted by $B_R = \{x \in \mathbb{R}^N : |x| \leq R\}$. For $x = (x_1, \ldots, x_N) \in B_R$ we define the functions

$$f(x) = \begin{cases} \dfrac{x_1^2 - x_2^2}{2|x|^2}\left(\dfrac{N+2}{(-\ln|x|)^{\frac{1}{2}}} + \dfrac{1}{2(-\ln|x|)^{\frac{3}{2}}}\right) & \text{if} \quad |x| \neq 0 \\ 0 & \text{if} \quad |x| = 0 \end{cases}$$

and

$$g(x) = (-\ln R)^{\frac{1}{2}}(x_2^2 - x_1^2).$$

Then

$$u(x) = \begin{cases} (x_2^2 - x_1^2)(-\ln|x|)^{\frac{1}{2}} & \text{if} \quad |x| \neq 0 \\ 0 & \text{if} \quad |x| = 0 \end{cases}$$

belongs to $C(\overline{B_R}) \cap C^\infty(\overline{B_R} \backslash \{0\})$ and satisfies

$$\begin{cases} -\Delta u = f(x) & \text{in} \quad B_R \backslash \{0\} \\ u = g & \text{on} \quad \partial B_R, \end{cases} \tag{2.1.2}$$

but u is not in $C^2(B_R)$, since

$$u_{x_1x_1}(x) = -2(-\ln|x|)^{\frac{1}{2}} + \frac{2x_1^2}{|x|^2(-\ln|x|)^{\frac{1}{2}}} + \frac{x_2^2 - x_1^2}{2|x|^2}\left[\frac{1}{(-\ln|x|)^{\frac{1}{2}}}\left(-1 + \frac{2x_1^2}{|x^2|}\right) - \frac{x_1^2}{|x|^2(-\ln|x|)^{\frac{3}{2}}}\right]$$

implies that $u_{x_1x_1}(x) \to \infty$ as $|x| \to 0$. However, it is possible that there is another solution of the equation that was of class C^2; to show that this does not happen, we will prove the following result, which is of interest in itself, since it gives a criterion to remove the singularity of harmonic functions.

Lemma 2.1.1. *Let $R > 0$ and u be a harmonic function in $B_R \backslash \{0\}$ that satisfies*

$$0 = \lim_{|x| \to 0} \begin{cases} \dfrac{u(x)}{\ln|x|} & \textit{if} \quad N = 2 \\ u(x)|x|^{N-2} & \textit{if} \quad N \geq 3. \end{cases}$$

Then u can be defined at 0 so that it is smooth and harmonic in B_R.

Proof. For simplicity, let us only consider the case $N \geq 3$. Since u is continuous on ∂B_R, there exists $v \in C^2(B_R)$ unique solution of Dirichlet problem

$$\begin{cases} -\Delta v = 0 & \text{in} \quad B_R \\ v = u & \text{on} \quad \partial B_R. \end{cases}$$

Using the continuity of v at 0 and the hypothesis, we have

$$\lim_{|x| \to 0} w(x)|x|^{N-2} = 0,$$

where $w(x) := u(x) - v(x)$. Moreover w is harmonic in $B_R \backslash \{0\}$ and $w(x) = 0$ on ∂B_R. For each $\varepsilon > 0$, set

$$w^+ = \frac{\varepsilon}{|x|^{N-2}} + w(x) \quad \text{and} \quad w^- = \frac{\varepsilon}{|x|^{N-2}} - w(x),$$

and choosing $\delta > 0$ small enough we have $w^+(x) > 0$ and $w^-(x) > 0$ on $|x| = \delta$. Applying the principle of the maximum to the region $\delta \leq |x| \leq R$, follows that

$$w^+(x) > 0 \quad \text{and} \quad w^-(x) > 0 \quad \text{in} \quad \delta \leq |x| \leq R.$$

In conclusion:

$$|w(x)| \leq \frac{\varepsilon}{|x|^{N-2}} \quad \text{in} \quad \delta \leq |x| \leq R,$$

for each $\varepsilon > 0$, from where, $w(x) \equiv 0$ if $x \neq 0$, since we can make the same argument with ρ as close to zero as necessary to encompass each point $x \neq 0$. Therefore u is equal to the harmonic function v in $B_R \backslash \{0\}$, from where defining $u(0) = v(0)$ the proof is conclude. □

Getting back to Problem **(2.1.2)**, assume there exists $v \in C^2(B_R)$ solution of **(2.1.2)**. Then $w(x) = u(x) - v(x)$ is harmonic in $B_R \backslash \{0\}$, continue in B_R and satisfies:

$$\lim_{|x| \to 0} w(x)|x|^{N-2} = 0.$$

Then, from **Lemma 2.1.1** w could be extended to a harmonic function in B_R. Thus, $w \in C^2(B_R)$ and therefore u must also belong to $C^2(B_R)$. Hence, $\lim_{|x| \to 0} u_{x_1 x_1}$ exists and we arrive at a contradiction.

Fortunately, if we hope obtain a C^2 solution, we must instead consider the Hölder Space $C^{0,\gamma}(\Omega)$ in place of $C(\Omega)$. In fact, from [25, Lemma 4.2] if ρ is bounded and locally Hölder continuous with exponent $0 < \gamma \leq 1$ in Ω , then the Newtonian potential of ρ, w, belongs to $C^2(\Omega)$ and satisfies $-\Delta w = \rho$ in Ω. On the other hand, if $\rho \in L^p(\Omega)$, $1 < p < \infty$, then from [25, Theorem 9.9] w is a strong solution, that is to say, $w \in W^{2,p}(\Omega)$ satisfies :

$$-\Delta u = \rho(x) \quad \text{a.e. in } \Omega.$$

Thus, since our approach considers $\rho \in L^\infty_{loc}(\mathbb{R}^N)$, we only expect to get solutions with regularly $C^{1,\gamma}$.

Now, since in the case Ω is the ball B_R, from [30, Theorem 12] we have an explicit formula for the solution of the problem:

$$\begin{cases} -\boldsymbol{\Delta u = \rho(x)} & \text{in} \quad \boldsymbol{B_R} \\ \boldsymbol{u = g} & \text{on} \quad \boldsymbol{\partial B_R,} \end{cases}$$

for given continuous functions ρ, g given by

$$u(x) = \int_{B_R} G_R(x,y)\rho(y)dy - \int_{\partial B_R} K(x,y)g(y)dS(y) \qquad \forall x \in B_R,$$

where K is the Poisson's Kernel (see [25, 2.29]) and G_R is the Green's function in B_R given by

$$\boldsymbol{G_R(x,y)} = \begin{cases} \boldsymbol{\Gamma(x-y) - \Gamma\left(\frac{|y|}{R}\left|x - \frac{R^2}{|y|^2}y\right|\right)} & \text{if} \quad \boldsymbol{y \neq 0} \\ \boldsymbol{\Gamma(x) - \Gamma(R)} & \text{if} \quad \boldsymbol{y = 0,} \end{cases}$$

for all $x \neq y$ in B_R, we will work with this representation based on the following properties of the Green's function in B_R.

Lemma 2.1.2. *The Green's function in B_R satisfies:*

i) If $R < R'$, then $G_R(x,y) < G_{R'}(x,y)$ for all $x \neq y$ in B_R, that is said that the Green's function in B_R is increasing with $R > 0$.

ii) $G_R(x,y) \to \Gamma(x-y)$ as $R \to \infty$ for all $x \neq y$ in $\mathbb{R}^N$.

The following result gives us a representation of the solutions of the Dirichlet problem:

$$\begin{cases} -\Delta u = \rho(x) & \text{in} \quad B_R \\ u = 0 & \text{on} \quad \partial B_R. \end{cases} \tag{2.1.3}$$

Lemma 2.1.3. *Let $\rho \in L^\infty_{loc}(\mathbb{R}^N)$, $\rho(x) \geq 0$ and ρ not identically zero. Then for each $R > 0$ the Dirichlet problem **(2.1.3)** has only one weak solution $u_R \in H^1_0(B_R)$, which is increasing with R. In addition*

$$u_R(x) = \int_{B_R} G_R(x,y)\rho(y)dy.$$

Proof. From [8] (see also [30]) Problem **(2.1.3)** has only one weak solution u_R obtained by

$$\min_{u \in H^1_0(B_R)} \left\{ \frac{1}{2}\int_{B_R} |\nabla u|^2 dx - \int_{B_R} \rho(x) u dx \right\},$$

and from [25, Theorem 8.8] follows that $u_R \in C^{1,\gamma}(B_R)$, for some $0 < \gamma < 1$. Since $\rho(x)$ is not identically zero we also have $u_R \geq 0$ in B_R and $u_R \neq 0$, even more, from [25, Theorem 8.19] (strong maximum principle for weak solutions), we have $u_R > 0$ in B_R.

Now, we claim that u_R is increasing with R, that is, if $R' > R$ then $u_{R'} \geq u_R$ in B_R. In fact, let $\varphi \in C^\infty_0(B_R)$ with $\varphi \geq 0$. Then, from Green's identities

$$\begin{aligned} -\int_{B_R} u_{R'}\Delta\varphi dx &\geq -\int_{B_{R'}} u_{R'}\Delta\varphi dx = \int_{B_{R'}} \nabla u_{R'}\nabla\varphi dx = \int_{B_{R'}} \rho(x)\varphi(x)dx \\ &\geq \int_{B_R} \rho(x)\varphi(x)dx = \int_{B_R} \nabla u_R \nabla\varphi dx \\ &= -\int_{B_R} u_R\Delta\varphi dx, \end{aligned}$$

from where

$$\int_{B_R} (u_{R'} - u_R)\Delta\varphi dx \leq 0.$$

Therefore, the maximum principle implies that $u_{R'} \geq u_R$ in B_R (see **Lemma 1.5.3**).
Next, we will show that is u_R is given by

$$u_R(x) = \int_{B_R} G_R(x,y)\rho(y)dy.$$

First of all, note that u_R is well defined, since if $k > 0$ is such that $B_R \subset B(x,k)$, then

$$\begin{aligned}\int_{B_R} G_R(x,y)\rho(y)dy &\le \int_{B_R} \Gamma(x,y)\rho(y)dy \\ &\le \frac{\|\rho\|_{L^\infty(B(x,k))}}{N(N-2)w_N} \int_{B(x,k)} \frac{1}{|x-y|^{N-2}} dy \\ &= \frac{\|\rho\|_{L^\infty(B(x,k))}}{N(N-2)w_N} \int_{B(0,k)} \frac{1}{|y|^{N-2}} dy \\ &= \frac{\|\rho\|_{L^\infty(B(x,k))}}{N(N-2)w_N} N w_N \int_0^k \frac{r^{N-1}}{r^{N-2}} dr \\ &= \frac{k^2}{2(N-2)} \|\rho\|_{L^\infty(B(x,k))}.\end{aligned}$$

On the other hand, for $R > 0$ fix $y \in B_R$ and for every $\varepsilon > 0$ we define $V_\varepsilon := B_R \backslash B(y,\varepsilon)$. Then, from monotone convergence theorem follows that

$$\lim_{\varepsilon \to 0} \int_{V_\varepsilon} \Gamma(x-y)\Delta\varphi(x)dx = \int_{B_R} \Gamma(x-y)\Delta\varphi(x)dx.$$

Now we will estimate the previous integral of the left side. For this let $\varphi \in C_0^\infty(B_R)$ with $\varphi \ge 0$. Then using Green's identities

$$\begin{aligned}\int_{V_\varepsilon} \Gamma(x-y)\Delta\varphi(x)dx &= \int_{V_\varepsilon} \big(\Gamma(x-y)\Delta\varphi(x) - \Delta\Gamma(x-y)\varphi(x)\big)dx \\ &= \int_{\partial V_\varepsilon} \left(\Gamma(x-y)\frac{\partial\varphi}{\partial\nu}(x) - \varphi(x)\frac{\partial\Gamma}{\partial\nu}(x-y)\right) dS(x) \\ &= \int_{\partial B(y,\varepsilon)} \left(\Gamma(x-y)\frac{\partial\varphi}{\partial\nu}(x) - \varphi(x)\frac{\partial\Gamma}{\partial\nu}(x-y)\right) dS(x).\end{aligned}$$

Again from Green's identities we have

$$\begin{aligned}\int_{\partial B(y,\varepsilon)} \Gamma(x-y)\frac{\partial\varphi}{\partial\nu}(x)dS(x) &= \frac{1}{N(N-2)w_N\varepsilon^{N-2}} \int_{\partial B(y,\varepsilon)} \frac{\partial\varphi}{\partial\nu}(x)dS(x) \\ &= \frac{1}{N(N-2)w_N\varepsilon^{N-2}} \int_{B(y,\varepsilon)} \Delta\varphi dx \\ &\le \frac{1}{N(N-2)w_N\varepsilon^{N-2}} \max_{B_R} |\Delta\varphi| w_N \varepsilon^N \\ &= \frac{\varepsilon^2}{N(N-2)} \max_{B_R} |\Delta\varphi| \to 0 \quad \text{as} \quad \varepsilon \to 0.\end{aligned}$$

Since, for every $x \in \partial B(y,\varepsilon)$ we have

$$\frac{\partial\Gamma}{\partial\nu}(x-y) = \nabla\Gamma(x-y)\cdot\nu = \left(\frac{-1}{Nw_N}\sum_{i=1}^N \frac{x_i-y_i}{|x-y|^N}\right)\cdot\left(-\frac{x_i-y_i}{|x-y|}\right) = \frac{1}{Nw_N\varepsilon^{N-1}}.$$

we obtain

$$\begin{aligned}\int_{\partial B(y,\varepsilon)} \varphi(x)\frac{\partial \Gamma}{\partial \nu}(x-y)dS(x) &= \frac{1}{Nw_N\varepsilon^{N-1}}\int_{\partial B(y,\varepsilon)} \varphi(x)dS(x)\\ &= \frac{1}{Nw_N\varepsilon^{N-1}}\int_{\partial B(y,\varepsilon)} \big(\varphi(x)-\varphi(y)\big)dS(x)+\varphi(y)\\ &\le \max_{|x-y|=\varepsilon}|\varphi(y)-\varphi(x)|+\varphi(y)\\ &\to \varphi(y) \quad \text{as} \quad \varepsilon\to 0.\end{aligned}$$

In conclusion:

$$-\varphi(y) = \int_{B_R}\Gamma(x-y)\Delta\varphi(x)dx.$$

Then using Green's identities and Fubini's theorem we get:

$$\begin{aligned}\int_{B_R}\nabla u_R(x)\nabla\varphi(x)dx &= -\int_{B_R} u_R(x)\Delta\varphi(x)dx\\ &= -\int_{B_R}\left(\int_{B_R} G_R(x,y)\rho(y)dy\right)\Delta\varphi(x)dx\\ &= -\int_{B_R}\rho(y)\left(\int_{B_R} G_R(x,y)\Delta\varphi(x)dx\right)dy\\ &= -\int_{B_R}\rho(y)\left(\int_{B_R}\Gamma(x-y)\Delta\varphi(x)dx\right)dy\\ &= \int_{B_R}\rho(y)\varphi(y)dy,\end{aligned}$$

where we obtain that u_R is weak solution of Problem **(2.1.3)**. □

After having finished referring to the equation in bounded domains and due to our approach, we will give some facts about of the Poisson's equation:

$$-\Delta u = \rho(x) \ \text{ in } \mathbb{R}^N. \qquad (\mathbf{P_e})$$

In what follows, N will be an integer greater than or equal to 3.
The following result says that the Newtonian potential of ρ, in whole space, belongs to $C^1(\mathbb{R}^N)$.

Lemma 2.1.4. *Let $\rho \in L^\infty_{loc}(\mathbb{R}^N)$, $\rho(x)\ge 0$ and ρ not identically zero. Assume that the Newtonian potential of ρ, given by*

$$w(x) = \int_{\mathbb{R}^N}\Gamma(x-y)\rho(y)dy,$$

belongs to $L^\infty\big(\mathbb{R}^N\big)$. Then $w\in C^1(\mathbb{R}^N)$ and for any $x\in\mathbb{R}^N$

$$D_i w(x) = \int_{\mathbb{R}^N} D_i\Gamma(x-y)\rho(y)dy \ \textit{for all } i=1,\dots,N.$$

Proof. For each $i = 1, \ldots, N$, we define

$$v(x) = \int_{\mathbb{R}^N} D_i\Gamma(x-y)\rho(y)dy.$$

Since

$$\begin{aligned}
\left|\int_{\mathbb{R}^N} D_i\Gamma(x-y)\rho(y)dy\right| &= \left|\int_{\mathbb{R}^N\setminus B(x,1)} D_i\Gamma(x-y)\rho(y)dy + \int_{B(x,1)} D_i\Gamma(x-y)\rho(y)dy\right| \\
&\leq \frac{1}{Nw_N}\int_{\mathbb{R}^N\setminus B(x,1)} \frac{\rho(y)}{|x-y|^{N-1}}dy + \frac{\|\rho\|_{L^\infty(B(x,1))}}{Nw_N}\int_{B(x,1)} \frac{1}{|x-y|^{N-1}}dy \\
&\leq \frac{1}{Nw_N}\int_{\mathbb{R}^N\setminus B(x,1)} \frac{\rho(y)}{|x-y|^{N-2}}dy + \frac{\|\rho\|_{L^\infty(B(x,1))}}{Nw_N}\int_{B(x,1)} \frac{1}{|x-y|^{N-1}}dy \\
&\leq (N-2)w(x) + \frac{\|\rho\|_{L^\infty(B(x,1))}}{Nw_N}\int_{B(0,1)} \frac{1}{|y|^{N-1}}dy \\
&= (N-2)w(x) + \|\rho\|_{L^\infty(B(x,1))},
\end{aligned}$$

follows that v_i is well defined. We now show that $v_i = D_i w_i$ for each $i = 1, \ldots, N$. To do so, for $\varepsilon > 0$, let $\eta_\varepsilon(x,y) = \eta(|x-y|/\varepsilon)$ where $\eta = \eta(|x|)$ is some nonnegative radial function in $C^1(\mathbb{R}^N)$ with $0 \leq \eta \leq 1$, $0 \leq \eta' \leq 2$ and

$$\eta(|x|) := \begin{cases} 0 & \text{if} \quad |x| \leq 1, \\ 1 & \text{if} \quad |x| \geq 2. \end{cases}$$

Define for $\varepsilon > 0$

$$w_\varepsilon(x) = \int_{\mathbb{R}^N} \eta_\varepsilon(x,y)\Gamma(x-y)\rho(y)dy.$$

Clearly, $w_\varepsilon \in C^1(\mathbb{R}^N)$ and

$$\begin{aligned}
|w(x) - w_\varepsilon(x)| &= \left|\int_{\mathbb{R}^N} \Gamma(x-y)\rho(y)dy - \int_{\mathbb{R}^N} \eta_\varepsilon(x,y)\Gamma(x-y)\rho(y)dy.\right| \\
&= \left|\int_{B(x,2\varepsilon)} (1-\eta_\varepsilon(x,y))\Gamma(x-y)\rho(y)dy\right| \\
&\leq \frac{\|\rho\|_{L^\infty(B(x,1))}}{N(N-2)w_N}\int_{B(x,2\varepsilon)} \frac{1}{|x-y|^{N-2}}dy \\
&= \frac{\|\rho\|_{L^\infty(B(x,1))}}{N(N-2)w_N}\int_{B(0,2\varepsilon)} \frac{1}{|y|^{N-2}}dy \\
&= \frac{2\varepsilon^2}{(N-2)}\|\rho\|_{L^\infty(B(x,1))}.
\end{aligned}$$

In the similar way and considering $\eta' \equiv 0$ on $\mathbb{R}^N \backslash B_{2\varepsilon}$, we have

$$\begin{aligned}
|v(x) - D_i w_\varepsilon(x)| &= \left| \int_{\mathbb{R}^N} D_i\Gamma(x-y)\rho(y)dy - \int_{\mathbb{R}^N} D_i\big(\eta_\varepsilon(x,y)\Gamma(x-y)\big)\rho(y)dy. \right| \\
&= \left| \int_{\mathbb{R}^N} \left((1-\eta_\varepsilon(x,y))D_i\Gamma(x-y) - \eta'_\varepsilon(x,y)\frac{x-y}{\varepsilon|x-y|}\Gamma(x-y) \right) \rho(y)dy \right| \\
&\le \|\rho\|_{L^\infty(B(x,1))} \int_{B(x,2\varepsilon)} \Big(|D_i\Gamma(x-y)| + \frac{2}{\varepsilon}|\Gamma(x-y)| \Big) dy \\
&\le \frac{\|\rho\|_{L^\infty(B(x,1))}}{nw_N} \int_{B(x,2\varepsilon)} \left(\frac{1}{|x-y|^{N-1}} + \frac{2}{\varepsilon(N-2)|x-y|^{N-2}} \right) dy \\
&= \|\rho\|_{L^\infty(B(x,1))} \left(2\varepsilon + \frac{1}{\varepsilon(N-2)} 4\varepsilon^2 \right) \\
&= \frac{2N\varepsilon}{N-2} \|\rho\|_{L^\infty(B(x,1))}.
\end{aligned}$$

In either case, we conclude that as $\varepsilon \to 0$, $w_\varepsilon(x) \to w(x)$ and $D_i w_\varepsilon(x) \to v_i(x)$ for every $x \in \mathbb{R}^N$. Therefore, $w \in C^1(\mathbb{R}^N)$ and $v_i = D_i w$, for each $i = 1, \ldots, N$.

□

Now we give the property (H) introduced by Brezis and Kamin [9], which will be used throughout all this work.

Definition 2.1.5.
Let $\rho \in L^\infty_{loc}(\mathbb{R}^N)$, $\rho(x) \ge 0$ and ρ not identically zero. We said that ρ has the property (H) if there exist a bounded solution of Poisson's equation **($\mathbf{P_e}$)**.

Remark 2.1.1. In order not to move away from the work of Breziz and Kamin, we have given the same definition introduced by them; however, the solutions in the above definition are actually weak solutions, therefore, when we refer to solutions of Problem **($\mathbf{P_e}$)**, we are actually assuming that they are weak solutions.

The next result gives a sufficient and necessary condition to have the property (H).

Lemma 2.1.6. *Let $\rho \in L^\infty_{loc}(\mathbb{R}^N)$, $\rho(x) \ge 0$ and ρ not identically zero. Then ρ satisfies property (H) iff*

$$\frac{c}{|x|^{N-2}} * \rho \in L^\infty\big(\mathbb{R}^N\big),$$

where we have denote by $c = (N(N-2)w_N)^{-1}$.

Proof. Suppose the property (H) is satisfied. Then, there exists U a bounded solution of **($\mathbf{P_e}$)**. By adding a constant we may always assume that $U \ge 0$ in $\mathbb{R}^N$. On the other hand, for each $R > 0$, from **Lemma 2.1.3**, Problem **(2.1.3)** has only one increasing weak solution $u_R \in H^1_0(B_R)$. In addition

$$u_R(x) = \int_{B_R} G_R(x,y)\rho(y)dy.$$

Let $\varphi \in C_0^\infty(B_R)$ with $\varphi \geq 0$. Then, from Green's identities

$$
\begin{aligned}
-\int_{B_R} U\Delta\varphi dx &= \int_{B_R} \nabla U \nabla \varphi dx = \int_{\mathbb{R}^N} \nabla U \nabla \varphi dx = \int_{\mathbb{R}^N} \rho(x)\varphi(x) dx \\
&\geq \int_{B_R} \rho(x)\varphi(x)dx = \int_{B_R} \nabla u_R \nabla \varphi dx \\
&= -\int_{B_R} u_R \Delta\varphi dx,
\end{aligned}
$$

from where

$$\int_{B_R} (U - u_R)\Delta\varphi dx \leq 0.$$

Therefore, the maximum principle implies that $u_R \leq U$ in B_R for all R. Then, using the monotone convergence theorem, we get

$$\lim_{R\to\infty} u_R(x) = \int_{\mathbb{R}^N} \Gamma(x-y)\rho(y)dy = \frac{c}{|x|^{N-2}} * \rho \in L^\infty\big(\mathbb{R}^N\big).$$

Reciprocally, suppose

$$u_\infty(x) := \frac{c}{|x|^{N-2}} * \rho \in L^\infty\big(\mathbb{R}^N\big).$$

From **Lemma 2.1.4**, we have $u_\infty \in C^1\big(\mathbb{R}^N\big)$. Now, let $\varphi \in C_0^\infty(\mathbb{R}^N)$ with $\varphi \geq 0$. Since u_R is a lower solution of $(\mathbf{P_R})$, for $R > 0$ large enough, using Green's identities, we have

$$
\begin{aligned}
\int_{B_R} \rho\varphi dx &= \int_{B_R} \nabla u_R \nabla\varphi dx = -\int_{B_R} u_R \Delta\varphi dx \\
&= -\int_{B_R} u_R\Big((\Delta\varphi)_+ - (\Delta\varphi)_-\Big)dx,
\end{aligned}
$$

from where, using monotone convergence theorem, follows that

$$\int_{\mathbb{R}^N} \rho\varphi dx = -\int_{\mathbb{R}^N} u_\infty \Delta\varphi dx = \int_{\mathbb{R}^N} \nabla u_\infty \nabla\varphi dx.$$

Therefore the function $u_\infty \in H^1\big(\mathbb{R}^N\big) \cap L^\infty\big(\mathbb{R}^N\big)$ provides a bounded weak solution of $(\mathbf{P_e})$, and as a consequence the lemma is proved.

□

The next result is a consequence of the previous theorem.

Corollary 2.1.7. *Suppose that ρ satisfies property (H). Then u_∞ is the minimal positive solution of $(\mathbf{P_e})$.*

Proof. From **Theorem 2.1.6**, since ρ satisfies property (H), follows that w_∞ is a bounded positive solution of $(\mathbf{P_e})$. Let U be a bounded positive solution of $(\mathbf{P_e})$. The maximum principle implies implies $u_R \leq U$ in B_R for all $R > 0$. Then

$$u(x) := \lim_{R\to\infty} u_R(x) \;\text{ exist for every } x \in \mathbb{R}^N,$$

and $u \leq U$ in $\mathbb{R}^N$. Since

$$\lim_{R\to\infty} u_R(x) = \int_{\mathbb{R}^N} \Gamma(x-y)\rho(y)dy = \frac{c}{|x|^{N-2}} * \rho = u_\infty(x),$$

follows that $u_\infty \leq U$ in $\mathbb{R}^N$. Therefore u_∞ is the minimal positive solution of **($\mathbf{P_e}$)**. □

The following result will allow us to show that u_∞ tends to zero at infinity in a sense that will be specified later.

Lemma 2.1.8. *Suppose that ρ satisfies property (H). Then*

$$\lim_{R\to\infty} \fint_{\partial B_R} u_\infty(y)dS(y) = 0.$$

Proof. By Fubini's theorem we have

$$\begin{aligned}
\frac{1}{R^{N-1}} \int_{\partial B_R} u_\infty(y)dS(y) &= \frac{1}{R^{N-1}} \int_{\mathbb{R}^N} \rho(x) \left(\int_{|y|=R} \frac{dS(y)}{|x-y|^{N-2}} \right) dx \\
&= \frac{1}{R^{N-1}} \left(\int_{|x|<R} \rho(x)I(x)dx + \int_{|x|>R} \rho(x)I(x)dx \right),
\end{aligned}$$

where we have denoted by

$$I(x) = \int_{|y|=R} \frac{dS(y)}{|x-y|^{N-2}}.$$

Let $y \in \partial B_R$. Since the function

$$x \mapsto \Phi(x) = \frac{1}{N(N-2)w_N} \frac{1}{|y-x|^{N-2}}$$

is harmonic for all $x \neq y$ in $\mathbb{R}^N$, we distinguish two cases:

i) $|x| < R$: By Mean-value formulas, for Laplace's equation (see [25], [30]), we have

$$\frac{1}{Nw_N R^{N-1}} \int_{\partial B(0,R)} \frac{1}{N(N-2)w_N} \frac{1}{|y-x|^{N-2}} dS(y) = \Phi(0) = \frac{1}{N(N-2)w_N} \frac{1}{|y|^{N-2}}$$

From where

$$\int_{|y|=R} \frac{dS(y)}{|y-x|^{N-2}} = Nw_R \frac{R^{N-1}}{R^{N-2}} = Nw_N R.$$

ii) $|x| > R$: We have

$$\begin{aligned}
|y-x|^2 &= |x|^2 - 2x \cdot y + |y|^2 \\
&= \frac{|y|^2}{R^2}|x|^2 - 2x \cdot y + R^2 \\
&= |x|^2 \left(\frac{|y|^2}{R^2} - 2y \cdot \frac{x}{|x|^2} + \frac{R^2}{|x|^2} \right) \\
&= |x|^2 \left(\frac{|y|^2}{R^2}|x|^2 - 2y \cdot \frac{x}{|x|^2} + R^2 \frac{|x|^2}{|x|^4} \right) \\
&= |x|^2 \left| \frac{y}{R} - R\frac{x}{|x|^2} \right|^2 \\
&= \left| \frac{x}{R} \right|^2 \left| y - R^2 \frac{x}{|x|^2} \right|^2.
\end{aligned}$$

Then

$$\int_{|y|=R} \frac{dS(y)}{|y-x|^{N-2}} = \left(\frac{R}{|x|}\right)^{N-2} \int_{\partial B_R} \frac{dS(y)}{\left|y - R^2 \frac{x}{|x|^2}\right|^{N-2}}.$$

However $\left|R^2 \frac{x}{|x|^2}\right| < R$. Then, from i) we have

$$\int_{|y|=R} \frac{dS(y)}{|y-x|^{N-2}} = \left(\frac{R}{|x|}\right)^{N-2} N w_N R.$$

Therefore from i), ii) we see that

$$\fint_{\partial B_R} u_\infty(y) dS(y) = \frac{c}{R^{N-2}} \int_{|x|<R} \rho(x)dx + c \int_{|x|>R} \frac{\rho(x)}{|x|^{N-2}} dx. \tag{2.1.4}$$

Using that $u_\infty \in L^\infty(\mathbb{R}^N)$ and dominated convergence theorem, as $R \to \infty$, the second integral of **(2.1.4)** tends to zero. We estimate the first one by

$$\frac{c}{R^{N-2}} \int_{|x|<R} \rho(x)dx = \frac{c}{R^{N-2}} \left(\int_{|x|<R_0} \rho(x)dx + \int_{R_0<|x|<R} \rho(x)dx \right),$$

for some $R_0 > 0$. To determine R_0 note that

$$\begin{aligned} \frac{1}{R^{N-2}} \int_{R_0<|x|<R} \rho(x)dx &\le \int_{R_0<|x|<R} \frac{\rho(x)}{|x|^{N-2}} dx \\ &\le \int_{R_0<|x|} \frac{\rho(x)}{|x|^{N-2}} dx, \end{aligned}$$

for each $R_0 > 0$. Thus, for $\varepsilon > 0$ we choose $R_0 > 0$ large enough satisfying

$$\int_{R_0<|x|} \frac{\rho(x)}{|x|^{N-2}} dx < \varepsilon.$$

Then, for this R_0 we choose $R > 0$ big enough so that:

$$\frac{1}{R^{N-2}} \int_{|x|<R_0} \rho(x)dx \le \frac{C R_0^{N-1}}{R^{N-2}} \|\rho\|_{L^\infty(B_{R_0})} < \varepsilon.$$

Therefore, from **(2.1.14)**, we get

$$\lim_{R\to\infty} \fint_{\partial B_R} u_\infty(y) dS(y) = 0.$$

□

Corollary 2.1.9. *Suppose that ρ satisfies property (H). Then*

$$\liminf_{|x|\to\infty} u_\infty(x) = 0.$$

Proof. Since $u_\infty(x) > 0$ for all $x \in \mathbb{R}^N$,

$$\liminf_{|x|\to\infty} u_\infty(x)$$

exists and is greater than or equal to 0. Now, suppose by contradiction that

$$\liminf_{|x|\to\infty} u_\infty(x) > 0.$$

Then, constants $C, M > 0$ would exists such that

$$u_\infty(y) \geq C > 0, \quad \forall |y| \geq M.$$

Thus, for $R > M$ we have

$$\int_{\partial B_R} u_\infty(y) dS(y) \geq C|\partial B_R| = CR^{N-1},$$

and consequently

$$⨍_{\partial B_R} u_\infty(y) dS(y) \geq C,$$

which contradicts the **Lemma 2.1.8** . Therefore

$$\liminf_{|x|\to\infty} u_\infty(x) = 0.$$

□

Lemma 2.1.10. *Suppose that ρ satisfies property (H). Then any bounded positive solution U of* ($\mathbf{P_e}$) *such that*

$$\liminf_{|x|\to\infty} U(x) = 0$$

coincides with u_∞.

Proof. Since u_∞ is the minimal positive solution of ($\mathbf{P_e}$) we have $u_\infty \leq U$ in $\mathbb{R}^N$ and

$$-\Delta(U - u_\infty) = 0 \;\text{ in } \mathbb{R}^N$$

holds in the weak sense, that is to say then for every :

$$\int_{\mathbb{R}^N} (U - u_\infty)\Delta\varphi dx = 0 \;\text{ for all } \varphi \in C_0^\infty(\mathbb{R}^N).$$

Then from [28, Corollary 1.2.1] (Weyl's lemma), follows that $(U - u_\infty)$ is harmonic in $\mathbb{R}^N$. Furthermore $(U - u_\infty)$ is bounded, then Liouville theorem yields

$$U - u_\infty = C \;\text{ for all } x \in \mathbb{R}^N,$$

for some constant $C \geq 0$. Using that

$$\liminf_{|x|\to\infty} \big(U - u_\infty\big)(x) = 0,$$

we get $C = 0$. Consequently $U = u_\infty$ in $\mathbb{R}^N$. □

Corollary 2.1.11. *Any bounded positive solution U of* ($\mathbf{P_e}$) *such that*

$$\lim_{R\to\infty} \fint_{\partial B_R} U(y)dS(y) = 0.$$

coincides with u_∞.

Proof. From **Corollary 2.1.9** proof, U satisfies

$$\liminf_{|x|\to\infty} U(x) = 0.$$

Then, by **Corollary 2.1.10**, $U = u_\infty$ in $\mathbb{R}^N$. □

Lemma 2.1.12. *Assume $U \in L^\infty(\mathbb{R}^N)$, with $\Delta U \in L^\infty_{loc}(\mathbb{R}^N)$, satisfies*

$$\begin{cases} -\boldsymbol{\Delta U} \leq \boldsymbol{\rho(x)} \ \text{in } \mathbb{R}^{\boldsymbol{N}}, \\ \displaystyle\lim_{\boldsymbol{R}\to\infty} \fint_{\boldsymbol{\partial B_R}} \boldsymbol{U(y)ds(y) = 0.} \end{cases}$$

Then $U \leq u_\infty$ in $\mathbb{R}^N$.

Proof. Set $g = -\Delta(u_\infty - U)$. Then $g \in L^\infty_{loc}(\mathbb{R}^N)$ and $u_\infty - U$ is a bounded solution of

$$-\Delta u = g(x) \text{ in } \mathbb{R}^N.$$

Thus, for every $\varphi \in C_0^\infty(\mathbb{R}^N)$ with $\varphi \geq 0$, we have

$$\int_{\mathbb{R}^N} g(x)\varphi dx = \int_{\mathbb{R}^N} \nabla (u_\infty - U) \nabla\varphi dx \geq 0.$$

Therefore $g(x) \geq 0$. Then using equality:

$$\lim_{R\to\infty} \fint_{\partial B_R} \big(u_\infty - U\big)(y)dS(y) = 0,$$

from **Corollary 2.1.11**, we obtain

$$u_\infty - U = \frac{c}{|x|^{N-2}} * g \geq 0.$$

Thus $U \leq u_\infty$ in $\mathbb{R}^N$.

□

Corollary 2.1.13. *Suppose that ρ_1 and ρ_2 they are satisfies property (H) and $\rho_1 \leq \rho_2$ in $\mathbb{R}^N$. Let U_1, U_2 be bounded positive solutions of* ($\mathbf{P_e}$)*, when $\rho = \rho_1$ and $\rho = \rho_2$, respectively, satisfying*

$$\lim_{R\to\infty} \fint_{\partial B_R} U_i(y)ds(y) = 0 \quad \text{for } i = 1, 2.$$

Then $U_1 \leq U_2$ in $\mathbb{R}^N$.

Proof. Since $U_1 \in L^\infty(\mathbb{R}^N)$ satisfies:

$$\begin{cases} -\boldsymbol{\Delta U_1 \leq \rho_2(x)} \text{ in } \mathbb{R}^{\boldsymbol{N}} \\ \lim\limits_{\boldsymbol{R\to\infty}} \boldsymbol{\fint_{\partial B_R} U_1(y)ds(y) = 0,} \end{cases}$$

from **Lemma 2.1.12**, we have

$$U_1 \leq \frac{c}{|x|^{N-2}} * \rho_2.$$

Then, using that

$$\lim_{R\to\infty} \fint_{\partial B_R} U_2(y)ds(y) = 0,$$

from **Corollary 2.1.11**, we see that $U_2 = \dfrac{c}{|x|^{N-2}} * \rho_2$. Therefore, we find that

$$U_1 \leq U_2 \ \text{ in } \ \mathbb{R}^N.$$

That is to say, any bounded positive solution of **($\mathbf{P_e}$)** that vanishing in infinity depends monotonically on ρ. □

The following proposition shows us a class of ρ that satisfies property (H), which will be used in later sections of this work.

Proposition 2.1.14. *Assume*

$$\rho(x) = \frac{1}{1+|x|^\beta} \quad \textit{for all } x \in \mathbb{R}^N.$$

i) If $\beta > 2$. Then ρ satisfies the property (H).

ii) If $\beta \leq 2$. Then ρ does not satisfy the property (H).

Proof. Let $r \geq 0$ and put $|x| = r$, then

$$\rho(r) = \frac{1}{1+|r|^\beta}.$$

Let

$$U(r) = \int_r^{+\infty} \left(s^{1-N} \int_0^s t^{N-1}\rho(t)dt \right) ds.$$

So, through simple calculations, we have

$$U'(r) = -r^{1-N} \int_0^r t^{N-1}\rho(t)dt,$$

$$U''(r) = -(1-N)r^{-N} \int_0^r t^{N-1}\rho(t)dt - \rho(r) = \frac{1-N}{r}U'(r) - \rho(r).$$

Therefore

$$-\Delta U(r) = -U''(r) - \frac{N-1}{r}U'(r) = \rho(r).$$

Next, we show that U is bounded when $\beta > 2$. Here we distinguish the cases in which $2 < \beta < N$ and $\beta \geq N$.

a) Let $2 < \beta < N$. If $r \geq 1$, it follows that

$$U(r) = \int_r^{+\infty} \left(s^{1-N} \int_0^s \frac{t^{N-1}}{1+t^\beta} dt \right) ds \leq \int_r^{+\infty} \left(s^{1-N} \int_0^s t^{N-1-\beta} dt \right) ds$$
$$= \frac{1}{N-\beta} \int_r^{+\infty} s^{1-\beta} ds$$
$$= \frac{1}{(N-\beta)(2-\beta)} \lim_{b \to +\infty} \left(b^{2-\beta} - r^{2-\beta} \right)$$
$$= \frac{r^{2-\beta}}{(N-\beta)(\beta-2)}$$
$$\leq \frac{1}{(N-\beta)(\beta-2)}.$$

Similarly, if $0 < r < 1$, U is shown and is bounded.

b) Let $\beta \geq N$. Here, using the next fact:
For every $a \in (0, N-2)$ there are $c_0 > 0$ and $r_1 \geq 1$ large enough such that $ln(s) \leq c_0 s^a$ for all $s \geq r_1$, follow that

$$ln\left(1+s^N\right) \leq ln\left(2s^N\right) = ln(2) + Nln(s) \leq ln(2) + c_0 N s^a \text{ for every } s \geq r_1.$$

Thus, if $r \geq r_1$, it follows that

$$\int_r^{+\infty} \left(s^{1-N} \int_0^s \frac{t^{N-1}}{1+t^\beta} dt \right) ds = \int_r^{+\infty} \left(s^{1-N} \int_0^1 \frac{t^{N-1}}{1+t^\beta} dt \right) ds + \int_r^{+\infty} \left(s^{1-N} \int_1^s \frac{t^{N-1}}{1+t^\beta} dt \right) ds$$
$$\leq \int_r^{+\infty} \left(s^{1-N} \int_0^1 t^{N-1} dt \right) ds + \int_r^{+\infty} \left(s^{1-N} \int_1^s \frac{t^{N-1}}{1+t^N} dt \right) ds$$
$$= \frac{1-\ln(2)}{N} \int_r^{+\infty} s^{1-N} ds + \frac{1}{N} \int_r^{+\infty} s^{1-N} \ln\left(1+s^N\right) ds$$
$$\leq \frac{1}{N} \int_r^{+\infty} s^{1-N} ds + c_0 \int_r^{+\infty} s^{1+a-N} ds$$
$$= \frac{r^{2-N}}{N(N-2)} + \frac{c_0 r^{2+a-N}}{N-2-a}$$
$$\leq \frac{(1+c_0) r^{2+a-N}}{N-2-a}$$
$$\leq \frac{1+c_0}{N-2-a}.$$

Similarly, if $0 < r < r_1$, U is shown is bounded.

Therefore, from a) and b), we conclude that ρ satisfies property (H). Now, with regard to the case $\beta \leq 2$, using inequality

$$\frac{2t^{N-1}}{1+t^\beta} \geq \begin{cases} t^{N-1} & \text{if } \beta \geq 0 \text{ and } 0 < t \leq 1 \text{ or } \beta < 0 \text{ and } t > 1 \\ t^{N-\beta-1} & \text{if } \beta \geq 0 \text{ and } t > 1 \text{ or } \beta < 0 \text{ and } 0 < t \leq 1, \end{cases}$$

for $0 \leq \beta \leq 2$, we see that

$$
\begin{aligned}
U(r) &= \int_r^{+\infty}\left(s^{1-N}\int_0^1 \frac{t^{N-1}}{1+t^\beta}dt\right)ds + \int_r^{+\infty}\left(s^{1-N}\int_1^s \frac{t^{N-1}}{1+t^\beta}dt\right)ds \\
&\geq \frac{1}{2}\int_r^{+\infty}\left(s^{1-N}\int_0^1 t^{N-1}dt\right)ds + \frac{1}{2}\int_r^{+\infty}\left(s^{1-N}\int_1^s t^{N-\beta-1}dt\right)ds \\
&= \frac{1}{2N}\int_r^{+\infty} s^{1-N}ds + \frac{1}{2(N-\beta)}\int_r^{+\infty} s^{1-N}\left(s^{N-\beta}-1\right)ds \\
&= \left(\frac{1}{2N}-\frac{1}{2(N-\beta)}\right)\int_r^{+\infty} s^{1-N}ds + \frac{1}{2(N-\beta)}\int_r^{+\infty} s^{1-\beta}ds \\
&= \left(\frac{1}{2N}-\frac{1}{2(N-\beta)}\right)\frac{r^{2-N}}{N-2} + \frac{1}{2(N-\beta)(2-\beta)}\lim_{b\to+\infty}\left(b^{2-\beta}-r^{2-\beta}\right) \\
&= +\infty.
\end{aligned}
$$

In the same way, it is shown that ρ does not satisfy property (H) when $\beta < 0$. □

Remark 2.1.2. The previous proposition tells us that Problem **(P_e)** has a classic solution, if and only if $\beta > 2$, when

$$\rho(x) = \frac{1}{1+|x|^\beta} \quad \text{for all } x \in \mathbb{R}^N.$$

Below we will give the main results presented in [9] regarding equation **(2.1.1)**, that is to say, of:

$$-\Delta u = \rho(x)u^\alpha \text{ in } \mathbb{R}^N,\ 0 < \alpha < 1 \ \text{ and } N \geq 3.$$

Theorem 2.1.15. *Problem **(2.1.1)** has a bounded positive solution if and only if ρ satisfies (H).*

Proof.

A. Sufficient condition.

First suppose ρ satisfies property (H). Let $R > 0$. We claim that for each R, the problem

$$\begin{cases} -\Delta u = \rho(x)u^\alpha & \text{in} \quad B_R \\ u = 0 & \text{on} \quad \partial B_R \end{cases} \tag{P_R}$$

possesses a unique positive weak solution $u_R \in H_0^1(B_R) \cap L^\infty(B_R)$.

Indeed, let φ_1 be a positive eigenfunction associated to the first eigenvalue $\overline{\lambda}_1$ of the equation

$$\begin{cases} -\Delta \varphi_1 = \overline{\lambda}_1 \rho(x)\varphi_1 & \text{in} \quad B_R \\ \varphi_1 = 0 & \text{on} \quad \partial B_R. \end{cases}$$

Since $0 < \alpha < 1$, we can take $\varepsilon > 0$ enough small satisfying

$$\overline{\lambda}_1 \leq \varepsilon^{\alpha-1}\varphi_1^{\alpha-1},$$

Then, for every $\phi \in C_0^\infty(B_R)$ with $\phi \geq 0$, we have

$$\int_{B_R} \nabla(\varepsilon\varphi_1)\nabla\phi dx = \int_{B_R} \varepsilon\overline{\lambda}_1\rho_1(x)\varphi_1\phi dx \leq \int_{B_R} \rho_1(x)(\varepsilon\varphi_1)^\alpha\phi dx$$

Therefore, $\varepsilon\varphi_1$ is a lower solution of **(P_R)**.
On the other hand, since ρ satisfies property (H), there exists U a bounded positive solution of **(P_e)**. Choosing $C \geq ||U||_\infty^{\alpha/(1-\alpha)}$ we have

$$C\rho(x) \geq \rho(x)(CU)^\alpha \text{ in } \mathbb{R}^N,$$

which implies that CU is an upper solution of **(P_R)** for all $R > 0$. Moreover we choose $\varepsilon > 0$ enough small such that $\varepsilon\varphi_1$ is a lower solution of **(P_R)** and $\varepsilon\varphi_1 \leq CU$. Therefore, **Theorem 1.5.2** give us the existence of a weak solution $u_R \in H_0^1(B_R) \cap L^\infty(B_R)$ of **(P_R)** such that

$$\varepsilon\varphi_1 \leq u_R \leq CU \text{ in } B_R.$$

Now, we will show that the solution u_R of **(P_R)** is unique. Indeed, suppose that u_1 and u_2 are two positive solutions of **(P_R)**. Define

$$S = \{s \in [0,1] : su_1 \leq u_2 \text{ on } B_R\}.$$

We observe that $0 \in S$. Furthermore, if $0 < s_0 \in S$, then for every $s \in (0, s_0)$ we have $s \in S$. Hence, $\eta u_1 \leq u_2$ on B_R where $\eta = \sup S$. We claim that $\eta = 1$. In fact, assume by contradiction that $\eta < 1$. Since $0 < \alpha < 1$, for every $\varphi \in C_0^\infty\big(B_R\big)$ with $\varphi \geq 0$ follows that

$$\begin{aligned}\int_{B_R} \nabla(u_2 - \eta u_1)\nabla\varphi dx &= \int_{B_R} \rho(x)\big(u_2^\alpha - \eta u_1^\alpha\big)\varphi dx \\ &\geq \int_{B_R} \rho(x)u_1^\alpha\big(\eta^\alpha - \eta\big)\varphi dx, \\ &\geq 0.\end{aligned}$$

Therefore, the following relation holds in the weak sense

$$\begin{cases} -\boldsymbol{\Delta(u_2 - \eta u_1) \geq 0} & \text{in } \boldsymbol{B_R} \\ \boldsymbol{u_2 - \eta u_1 = 0} & \text{on } \boldsymbol{\partial B_R}. \end{cases}$$

Then, using the Maximum principle and [38, Theorem 1] (Hopf's Lemma for weak solutions) we see that either

i) $u_2 - \eta u_1 > 0$ in B_R with $\dfrac{\partial}{\partial\nu}(u_2 - \eta u_1) < 0$ on ∂B_R, or

ii) $u_2 - \eta u_1 \equiv 0$.

In the first case, there would be some $\varepsilon_1 > 0$ such that $u_2 - \eta u_1 \geq \varepsilon_1 u_1$, that is to say $u_2 \geq (\eta + \varepsilon_1)u_1$, which is impossible. Respect ii), this case is also impossible, since if we would have $u_2 - \eta u_1 \equiv 0$, then for each $\varphi \in C_0^\infty(\mathbb{R}^N)$ with $\varphi \geq 0$ follows that

$$\begin{aligned} 0 &= \int_{B_R} \nabla(u_2 - \eta u_1)\nabla\varphi dx = \int_{B_R} \nabla u_2 \nabla\varphi dx - \eta \int_{B_R} \nabla u_1 \nabla\varphi dx \\ &= \int_{B_R} \rho(x) u_2^\alpha \varphi dx + \eta \int_{B_R} \rho(x) u_1^\alpha \varphi dx \\ &= (\eta^\alpha - \eta) \int_{B_R} \rho(x) u_1^\alpha \varphi dx, \end{aligned}$$

from where, using that ρ is not identically zero, $\eta = \eta^\alpha$ is obtained, which is impossible. Thus, we conclude that $\eta = 1$, i.e., $u_2 \geq u_1$. Similarly, $u_2 \leq u_1$. Consequently, $u_2 = u_1$. Therefore, the solution u_R of **(P_R)** is unique.

Next, we will prove that the sequence u_R is increasing with R. Indeed, let $R' > R$. Then, $u_{R'}$ is an upper solution of **(P_R)**. On the other hand, we can choose $\varepsilon > 0$ enough small for $\varepsilon\varphi_1$ to be a solution of **(P_R)** with $\varepsilon\varphi_1 \leq u_{R'}$ in B_R. This implies that there is a weak solution v of **(P_R)** with

$$\varepsilon\varphi_1 \leq v \leq u_{R'} \ \text{ in } B_R.$$

Since Problem **(P_R)** has only one weak solution given by u_R, follows that

$$u_R \leq u_{R'} \text{ in } B_R \text{ for } R' > R.$$

Now, we will prove the existence of a solution of problem **(2.1.1)**. In fact, since

$$u_R \leq CU \quad \text{in} \quad B_R, \tag{2.1.5}$$

for all $R > 0$ and u_R is increasing, we get

$$u(x) := \lim_{R\to\infty} u_R(x) \ \text{ exist for every } \ x \in \mathbb{R}^N,$$

and also

$$u \leq CU \quad \text{in} \quad \mathbb{R}^N. \tag{2.1.6}$$

Next, we will prove that u is a positive weak solution of **(2.1.1)**.

From **Lemma 2.1.3**, follows that

$$u_R(x) = \int_{B_R} G_R(x,y)\rho(y)u_R^\alpha(y)dy, \ x \in B_R.$$

Since G_R and u_R are increasing in R, using monotone convergence, and from **(2.1.6)**, we get

$$u(x) = c\int_{\mathbb{R}^N} \frac{\rho(y)u^\alpha(u)}{|x-y|^{N-2}}dy = \frac{c}{|x|^{N-2}} * \rho u^\alpha \in L^\infty(\mathbb{R}^N).$$

Thus, from **Lemma 2.1.6** the function u is a weak solution of

$$-\Delta u = \rho(x)u^\alpha \ \text{ in } \mathbb{R}^N.$$

Finally u is a minimal positive solution of **(2.1.1)**, since each bounded positive solution v of **(2.1.1)** is an upper solution of **(P_R)** and we can take $\varepsilon > 0$ small enough such that $\varepsilon\varphi_1$ is a lower solution of **(P_R)**, then we see that $u_R \leq v$ in B_R. Hence, letting $R \to \infty$, we conclude that $u \leq v$ in $\mathbb{R}^N$.

B. Necessary Condition.
Suppose u is a bounded positive solution of **(2.1.1)** and for each $R > 0$, $w_R \in H_0^1(B_R)$ is the only one weak solution of:

$$\begin{cases} \boldsymbol{-\Delta u = \rho(x)} & \text{in} \quad \boldsymbol{B_R} \\ \boldsymbol{u = 0} & \text{on} \quad \boldsymbol{\partial B_R,} \end{cases}$$

which is increasing with $R > 0$ and given by

$$w_R(x) = \int_{B_R} G_R(x,y)\rho(y)dy. \tag{2.1.7}$$

On the other hand, set

$$v = \frac{1}{1-\alpha}u^{1-\alpha}.$$

For each $\varphi \in C_0^\infty(\mathbb{R}^N)$ with $\varphi \geq 0$, the function $\psi = u^{-\alpha}\varphi$ is well defined and belong to $H^1(\mathbb{R}^N)$. Then, using ψ as a test function for the equation **(2.1.1)**, we have

$$\int_{\mathbb{R}^N} \nabla u \nabla \psi dx = \int_{\mathbb{R}^N} \rho(x) u^\alpha \psi dx,$$

where we get:

$$\begin{aligned} \int_{\mathbb{R}^N} \rho(x)\varphi dx &= \int_{\mathbb{R}^N} \nabla u \nabla \left(u^{-\alpha}\varphi\right) dx \\ &= \int_{\mathbb{R}^N} \nabla u \big(u^{-\alpha}\nabla\varphi - \alpha\varphi u^{-\alpha-1}\nabla u\big) dx \\ &= \int_{\mathbb{R}^N} \big(\nabla v \nabla \varphi - \alpha u^{-\alpha-1}|\nabla u|^2\varphi\big) dx \\ &\leq \int_{\mathbb{R}^N} \nabla v \nabla \varphi dx. \end{aligned}$$

Thus, from maximum principle, we have $w_R \leq v$ in B_R for all $R > 0$. Therefore

$$U(x) := \lim_{R\to\infty} w_R(x) \ \text{ exist for every } \ x \in \mathbb{R}^N$$

and

$$U(x) \leq v \text{ in } \mathbb{R}^N.$$

Therefore, in **(2.1.7)**, letting $R \to \infty$, from monotone convergence theorem, we obtain

$$U(x) = c\int_{\mathbb{R}^N} \frac{\rho(y)}{|x-y|^{N-2}}dy = \frac{c}{|x|^{N-2}} * \rho \in L^\infty(\mathbb{R}^N).$$

Then, from **Lemma 2.1.6**, we conclude that U is a weak solution of **($\mathbf{P_e}$)**. Furthermore, the next inequality holds:

$$\big((1-\alpha)U\big)^{\frac{1}{1-\alpha}} \leq u \ \text{ in } \ \mathbb{R}^N.$$

□

Remark 2.1.3. Since u satisfies **(2.1.6)** for any positive solution U of **(P_e)**; in particular we can take $U = u_\infty$ and by **corollary 2.1.9** we conclude that

$$\liminf_{|x|\to\infty} u(x) = 0.$$

Next, we will show a uniqueness result.

Theorem 2.1.16. *Assuming ρ has property (H), then there is exactly one bounded positive solution, u, of **(2.1.1)** satisfying*

$$\liminf_{|x|\to\infty} u(x) = 0. \tag{2.1.8}$$

Proof. This proof is divided into 3 steps:

Step 1.
Assuming $\rho_1 \leq \rho_2$ and they satisfy property (H). We claim that given any bounded positive solution u_1 of

$$\begin{cases} -\Delta u_1 = \rho_1(x) u_1^\alpha & \text{in} \quad \mathbb{R}^N \\ \lim\limits_{R\to\infty} \fint_{\partial B_R} u_1 = 0, \end{cases} \tag{2.1.9}$$

there exists a bounded positive solution u_2 of

$$\begin{cases} -\Delta u_2 = \rho_2(x) u_2^\alpha & \text{in} \quad \mathbb{R}^N \\ \lim\limits_{R\to\infty} \fint_{\partial B_R} u_2 = 0 \end{cases} \tag{2.1.10}$$

such that $u_1 \leq u_2$.
Indeed, since ρ_2 satisfies property (H) there exists, v, bounded positive weak solution of

$$-\Delta v = \rho_2(x) \quad \text{in } \mathbb{R}^N$$

vanishing into infinity.

Since $0 < \alpha < 1$, v and u_1 they are bounded, there exists $C > 0$ large enough such that

$$u_1^\alpha(x) \leq C \quad \text{and} \quad C^{\alpha-1} v^\alpha(x) \leq 1.$$

This implies that the following relation holds in the weak sense

$$-\Delta u_1 \leq C\rho_2(x) \quad \text{in } \mathbb{R}^N$$

and from **Lemma 2.1.12**, we have $u_1 \leq Cv$.
Put $v_1 = Cv$. Since v_1 is bounded, $\rho_2 v_1^\alpha$ also satisfy property (H) and consequently there exists, v_2, unique bounded positive weak solution of

$$\begin{cases} -\Delta v_2 = \rho_2(x) v_1^\alpha(x) & \text{in} \quad \mathbb{R}^N \\ \lim\limits_{R\to\infty} \fint_{\partial B_R} v_2 = 0. \end{cases}$$

Thus, we see that

$$-\Delta v_2 \leq C\rho_2(x) \quad \text{in } \mathbb{R}^N$$

holds in the weak sense. Then, from **Lemma 2.1.12** we also have $v_2 \leq Cv$. Moreover, since $u_1 \leq v_1$, we have

$$-\Delta u_1 \leq \rho_2(x) v_1^\alpha \quad \text{in } \mathbb{R}^N$$

in the weak sense, and again from **Lemma 2.1.12**, we get $u_1 \leq v_2$. In this way there exists a sequence v_n of bounded positive weak solutions of

$$\begin{cases} -\Delta v_n = \rho_2(x) v_{n-1}^\alpha(x) & \text{in} \quad \mathbb{R}^N \\ \lim\limits_{R\to\infty} \fint_{\partial B_R} v_n = 0 \end{cases}$$

satisfying

$$u_1 \leq \cdots \leq v_n \leq \cdots \leq v_2 \leq v_1 \text{ in } \mathbb{R}^N.$$

Therefore, there exists

$$\lim_{n\to\infty} v_n(x) := w(x) \text{ for every } x \in \mathbb{R}^N$$

and w is a bounded positive solution of **(2.1.10)** satisfying $u_1 \leq w$.

Step 2.
Assume we have proven uniqueness for any $\rho > 0$, then we also have uniqueness for a general $\rho \geq 0$.
For this purpose, let $\rho_\varepsilon = \rho + \varepsilon h$ where $h \in C^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ with $h > 0$. Also let u_ε be the unique bounded positive weak solution of

$$\begin{cases} -\Delta u_\varepsilon = \rho_\varepsilon(x) u_\varepsilon^\alpha & \text{in} \quad \mathbb{R}^N \\ \lim\limits_{R\to\infty} \fint_{\partial B_R} u_\varepsilon = 0, \end{cases}$$

and u be any bounded positive weak solution of

$$\begin{cases} -\Delta u = \rho(x) u^\alpha & \text{in} \quad \mathbb{R}^N \\ \lim\limits_{R\to\infty} \fint_{\partial B_R} u = 0. \end{cases}$$

By **step 1** and by the uniqueness of u_ε we have

$$u \leq u_\varepsilon.$$

Now, we prove that $u = v$, where v is the minimal positive weak solution of **(2.1.1)**, constructed in **Theorem 2.1.15**, vanishing in infinity.
Indeed let $u_{\varepsilon,R}$ and u_R be the positive weak solutions of

$$\begin{cases} -\Delta u_{\varepsilon,R} = \rho_\varepsilon(x) u_{\varepsilon,R}^\alpha & \text{in} \quad B_R \\ u_{\varepsilon,R} = 0 & \text{on} \quad \partial B_R \end{cases} \tag{2.1.11}$$

and

$$\begin{cases} -\Delta u_R = \rho(x) u_R^\alpha & \text{in} \quad B_R, \\ u_R = 0 & \text{on} \quad \partial B_R. \end{cases} \tag{2.1.12}$$

Multiply **(2.1.11)** by u_R and **(2.1.12)** by $u_{\varepsilon,R}$ and by integrating we have

$$\begin{aligned}\int_{B_R} \rho(x) u_{\varepsilon,R}^{\alpha} u_R^{\alpha}\big(u_{\varepsilon,R}^{1-\alpha} - u_R^{1-\alpha}\big)dx &= \int_{B_R} \rho(x)\big(u_{\varepsilon,R} u_R^{\alpha} - u_{\varepsilon,R}^{\alpha} u_R\big)dx \\ &= \int_{B_R} \big(\rho_\varepsilon(x) u_{\varepsilon,R}^{\alpha} u_R - \rho(x) u_{\varepsilon,R}^{\alpha} u_R\big)dx \\ &= \int_{B_R} \big(\rho_\varepsilon(x) - \rho(x)\big) u_{\varepsilon,R}^{\alpha} u_R dx \\ &= \varepsilon \int_{B_R} h u_{\varepsilon,R}^{\alpha} u_R dx \\ &\leq \varepsilon C ||h||_{L^1(\mathbb{R}^N)},\end{aligned}$$

where C is independent of R. Passing to the limit as $R \to \infty$ we obtain

$$\int_{\mathbb{R}^N} \rho(x) u_\varepsilon^{\alpha} v^{\alpha}\big(u_\varepsilon^{1-\alpha} - v^{1-\alpha}\big)dx \leq C\varepsilon$$

and considering $v \leq u \leq u_\varepsilon$, $0 < \alpha < 1$, we have

$$\int_{\mathbb{R}^N} \rho(x) u^{\alpha} v^{\alpha}(u^{1-\alpha} - v^{1-\alpha})dx \leq C\varepsilon.$$

Doing $\varepsilon \to 0$ we conclude

$$\int_{\mathbb{R}^N} \rho(x)\big(uv^{\alpha} - u^{\alpha} v\big)dx = 0.$$

Again, using $v \leq u$ we get

$$\int_{\mathbb{R}^N} \rho(x)\big(vv^{\alpha} - u^{\alpha} v\big)dx = 0.$$

Thus $\rho(x)v^{\alpha} = \rho(x)u^{\alpha}$. Hence, $\Delta(u - v) = 0$ in $\mathbb{R}^N$ in the sense weak. Finally using $\liminf_{|x|\to\infty}(u - v) = 0$, we conclude that

$$u = v \quad \text{in } \mathbb{R}^N \quad (\text{ see } \textbf{Lemma 2.1.10}).$$

Before going to the last stage, we must give a result about bounded domains, which involves the use of parabolic equations. For this, we begin by giving the following definition:

Definition 2.1.17. Let Ω the exterior of an $(n-1)$-dimensional smooth closed surface $\partial\Omega$ in the space $\mathbb{R}^N$, $N \geq 3$. Let $T > 0$, $m > 1$ and $w_0 \in L^\infty(\Omega)$. We said $w \in L^\infty_{loc}(\Omega \times (0,T))$ is a weak solution of the Filtration equation

$$\begin{cases} \rho(x)\dfrac{\partial w}{\partial t} - \Delta w^m = 0 & \text{in} \quad \Omega \times (0,T) \\ w(x,0) = w_0(x) & \text{in} \quad \Omega \\ w(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T), \end{cases} \tag{2.1.13}$$

if

$$\int_{\Omega\times(0,T)} \Big(w^m(x,t)\Delta\varphi(x,t) + \rho(x)w(x,t)\varphi_t(x,t)\Big)dxdt + \int_\Omega \rho(x)w_0(x)\varphi(x,0)dx = 0,$$

for all $\varphi \in C_0^\infty(\Omega \times (0,T))$.

Remark 2.1.4.

i) Here, consider the space $C_0^\infty(\Omega \times (0,T))$ be a set of functions $\varphi(x,t)$ belonging to $C^\infty(\Omega \times (0,T)) \cap C(\overline{\Omega} \times (0,T))$, such that

$$\varphi(x,t) = 0 \text{ on } \partial\Omega \times (0,T), \quad \varphi(x,t) = 0 \text{ on } \partial B_R \times (0,T),$$

and $\varphi = 0$ if $t > T - \varepsilon(\varphi)$ or $|x| > R(\varphi)$, where $\varepsilon(\varphi) \in (0,T)$, $R(\varphi) > R_0$ and $R_0 > 0$ is chosen so that the ball B_{R_0} contains the surface $\partial\Omega$. For more details see [17].

ii) Definition of weak solution of Problem (2.1.13), when Ω is the whole space $\mathbb{R}^N$, it is the same as the previous one, taking into account that $\partial\Omega = \emptyset$.

Theorem 2.1.18. *Let $N \geq 3$.*

a) If

$$\int_\Omega \frac{\rho(x)w_0(x)}{|x|^{N-2}}dx < \infty,$$

then, Problem (2.1.13) has only one weak solution, w, satisfying

$$\lim_{R\to\infty} R^{1-N} \int_{\partial B_R} \left(\int_0^T w^m(x,t)dt \right) dS(x) = 0.$$

b) If $w_0 \in C^{1,\alpha}(B_R)$, then $\partial w_R^m / \partial x_i$ exists and is continuous as a function of x_i everywhere in $B_R \times (0,T)$, for each $i = 1, \ldots, N$.

c) If there exists another solution $\tilde{w}(x,t)$ of (2.1.13) with $\tilde{w}(x,t) \geq 0$ on $\Omega \times (0,T)$ and $\tilde{w}(x,0) \geq w_0(x)$ then $\tilde{w}(x,t) \geq w(x,t)$ in $\Omega \times (0,T)$.

Proof.

a) The proof can be found in [17, Theorem 2].

b) The proof of this regularity result can be found in [4, Theorem].

c) The proof of this comparison result can be found in [17, Theorem 3].

□

Now, notice that if $u(x)$ is a bounded weak solution of (2.1.1). Then, for each $\tau > 0$,

$$z_\tau(x,t) = \frac{Cu^{\frac{1}{m}}}{(t+\tau)^{\frac{1}{m-1}}}$$

satisfies:

$$\frac{\partial z_\tau}{\partial t} = -\frac{Cu^{\frac{1}{m}}}{(m-1)(t+\tau)^{\frac{m}{m-1}}} \text{ and } z_\tau^m = \frac{C^m u}{(t+\tau)^{\frac{m}{m-1}}},$$

where $m = \frac{1}{\alpha} > 1$, put $C = (m-1)^{\frac{-1}{m-1}}$. From where, using that u is a weak solution of (2.1.1) and Green's identities, for each $\varphi \in C_0^\infty(\mathbb{R}^N \times (0,\infty))$ we have

$$\begin{aligned}
0 &= -\int_{\mathbb{R}^N\times(0,\infty)} \frac{\rho(x)u^{\frac{1}{m}}\varphi}{(t+\tau)^{\frac{m}{m-1}}}\left(C^m - \frac{C}{m-1}\right)dxdt \\
&= -\int_{\mathbb{R}^N\times(0,\infty)} \left(\frac{C^m}{(t+\tau)^{\frac{m}{m-1}}}\nabla u\nabla\varphi - \frac{Cu^{\frac{1}{m}}}{(m-1)(t+\tau)^{\frac{m}{m-1}}}\rho(x)\varphi\right)dxdt \\
&= -\int_{\mathbb{R}^N\times(0,\infty)} \left(\nabla z_\tau^m\nabla\varphi + \rho(x)\frac{\partial z_\tau}{\partial t}\varphi\right)dxdt \\
&= \int_{\mathbb{R}^N\times(0,\infty)} \left(z_\tau^m\Delta\varphi + \rho(x)z_\tau\varphi_t\right)dxdt + \int_{\mathbb{R}^N}\rho(x)z_\tau(x,0)\varphi(x,0)dx,
\end{aligned}$$

in other words, z_τ is a weak solution of the problem:

$$\begin{cases} \rho(x)\dfrac{\partial z_\tau}{\partial t} - \Delta z_\tau^m = 0 & \text{in} \quad \mathbb{R}^N\times(0,\infty) \\ z_\tau(x,0) = \dfrac{Cu^{\frac{1}{m}}}{\tau^{\frac{1}{m-1}}} & \text{in} \quad \mathbb{R}^N. \end{cases}$$

Furthermore, since $u \in C^{1,\alpha}(\mathbb{R}^N)$, from regularity $\partial w_R^m/\partial x$ is continuous as a function of x everywhere in $\mathbb{R}^N \times (0,T)$.
Next we will show the general result of uniqueness.

Step 3.
By **step 2**, we suppose that $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ and $\rho > 0$.
Let v be the minimal bounded positive weak solution of (2.1.1), constructed in **Theorem 2.1.15**, vanishing at infinity and let u be any bounded positive weak solution of (2.1.1) satisfying (2.1.8). Next we will prove that $u = v$.
In fact, let w_R be a only one weak solution of

$$\begin{cases} \rho(x)\dfrac{\partial w_R}{\partial t} - \Delta w_R^m = 0 & \text{in} \quad B_R\times(0,T) \\ w_R(x,0) = Cu^{\frac{1}{m}}(x) & \text{in} \quad B_R \\ w_R(x,t) = 0 & \text{on} \quad \partial B_R\times(0,T). \end{cases}$$

From comparison result, it follows that w_R is increasing with $R > 0$, and since $z_1(x,0) = Cu^{\frac{1}{m}}(x)$, we have

$$w_R(x,t) \le z_1(x,t) \text{ in } B_R\times(0,T),$$

for all $R > 0$. From where, letting $R \to \infty$, the sequence w_R increases to some limit $w_\infty(x,t)$ which satisfies

$$\begin{cases} \rho(x)\dfrac{\partial w_\infty}{\partial t} - \Delta w_\infty^m = 0 & \text{in} \quad \mathbb{R}^N\times(0,T) \\ w_\infty(x,0) = Cu^{\frac{1}{m}}(x) & \text{in} \quad \mathbb{R}^N \end{cases} \tag{2.1.14}$$

and also

$$w_\infty(x,t) \le z_1(x,t) \text{ in } \mathbb{R}^N\times(0,T).$$

Furthemore, we claim that $w_\infty(x,t) = z_1(x,t)$. For this purpose, notice that z_1 and w_∞ are weak solutions of **(2.1.14)**, then for each $\varphi \in C_0^\infty(\mathbb{R}^N \times (0,\infty))$ we have

$$0 = \int_{\mathbb{R}^N\times(0,\infty)} \left(\nabla(z_1^m - w_\infty^m)\nabla\phi + \rho(x)\frac{\partial}{\partial t}(z_1 - w_\infty)\phi\right) dxdt.$$

Now, let $\eta = \mathbb{R} \to \mathbb{R}$ a nonnegative function in $C^\infty(\mathbb{R})$ with $0 \le \eta \le 1$, $\eta' \le 0$,

$$\eta(s) := \begin{cases} 0 & \text{if} \quad s \ge 1, \\ 1 & \text{if} \quad s \le 0, \end{cases}$$

and for $\varepsilon \in (0,T)$, $\tau \in (0, T-\varepsilon)$, set

$$\eta_{\varepsilon\tau}(t) = \eta\left(\frac{t-\tau}{T-\varepsilon-\tau}\right).$$

Moreover, put

$$\phi(x) = \frac{1}{N(N-2)w_N}\left(\frac{1}{|x|^{N-2}} - \frac{1}{R^{N-2}}\right), \quad x \neq 0 \text{ in } B_R.$$

Thus, using the function $\varphi \in C_0^\infty(B_R \times (0,T))$ given by $\varphi = \phi \cdot \eta_{\varepsilon\tau}$ as test function, we find:

$$I_1 + I_2 := \int_{B_R\times(0,T)} \Big(\nabla(z_1^m - w_\infty^m)\nabla\varphi + \rho(x)\frac{\partial}{\partial t}(z_1 - w_\infty)\varphi\Big) dxdt = 0, \qquad \textbf{(2.1.15)}$$

for all $R > 0$. For $\varepsilon \in (0,T)$ small enough, an integration yields:

$$\begin{aligned} I_2 = \int_{B_R\times(0,T)} \rho(x)\frac{\partial}{\partial t}(z_1 - w_\infty)\varphi dxdt &= \int_{B_R\times(0,T)} \rho(x)\phi(x)\int_0^T \left(\frac{\partial}{\partial t}(z_1 - w_\infty)\eta_{\varepsilon\tau}(t)dt\right) dx \\ &= \int_{B_R} \rho(x)\phi(x)\Big(z_1(x,T) - w_\infty(x,T)\Big)dx. \end{aligned}$$

On the other hand, using Green's identities follows that

$$\int_{B_R} \nabla(z_1^m - w_\infty^m)\nabla\phi dx = -\int_{B_R}(z_1^m - w_\infty^m)\Delta\phi dx + \int_{\partial B_R}(z_1^m - w_\infty^m)\frac{\partial\phi}{\partial\nu}dS(x). \ \textbf{(2.1.16)}$$

Now, for every $\delta > 0$, we have

$$\begin{aligned} \int_{B_R}(z_1^m - w_\infty^m)\Delta\phi dx &= \int_{B_R\setminus B_\delta}(z_1^m - w_\infty^m)\Delta\phi dx + \int_{B_\delta}(z_1^m - w_\infty^m)\Delta\phi dx \\ &= \int_{B_\delta}(z_1^m - w_\infty^m)\Delta\phi dx \\ &= \int_{\partial B_\delta}(z_1^m - w_\infty^m)\frac{\partial\phi}{\partial\nu}dS(x) - \int_{B_\delta}\nabla(z_1^m - w_\infty^m)\nabla\phi dx \\ &:= I_3 - I_4. \end{aligned}$$

Regarding I_3, for every $x \in \partial B(y,\delta)$ we have

$$\frac{\partial \phi}{\partial \nu}(x) = \nabla\phi(x)\cdot\nu = \left(\frac{-1}{Nw_N}\sum_{i=1}^{N}\frac{x_i}{|x|^N}\right)\cdot\left(\frac{x_i}{|x|}\right) = -\frac{1}{Nw_N\delta^{N-1}}.$$

Therefore

$$\begin{aligned} I_3 = \int_{\partial B_\delta}(z_1^m - w_\infty^m)\frac{\partial\phi}{\partial\nu}dS(x) &= -\frac{1}{Nw_N\delta^{N-1}}\int_{\partial B_\delta}\big(z_1^m(x,t) - w_\infty^m(x,t)\big)dS(x)\\ &\to -\Big(z_1^m(0,t) - w_\infty^m(0,t)\Big) \text{ as } \delta\to 0. \end{aligned}$$

Now, regarding I_4, from regularity

$$\begin{aligned} I_4 = \left|\int_{B_\delta}\nabla(z_1^m - w_\infty^m)\nabla\phi dx\right| &\le \frac{\sup_{B_\delta}|\nabla(z_1^m - w_\infty^m)|}{Nw_N}\int_{B_\delta}\frac{1}{|x|^{N-1}}dx\\ &= \delta\sup_{B_\delta}|\nabla(z_1^m - w_\infty^m)|\\ &\to 0 \text{ as } \delta\to 0. \end{aligned}$$

Therefore for $\varepsilon\in(0,T)$ small enough, from **(2.1.16)**, we conclude that:

$$I_1 = \int_0^T\Big(z_1^m(0,t) - w_\infty^m(0,t)\Big)dt + \int_{\partial B_R\times(0,T)}(z_1^m - w_\infty^m)\frac{\partial\phi}{\partial\nu}dS(x)dt.$$

Next, we will show that the previous integral on the right side converges to 0, as $R\to\infty$, indeed, in a similar way to I_3, from **Lemma 2.1.8**, we have:

$$\begin{aligned} \left|\int_{\partial B_R\times(0,T)}(z_1^m - w_\infty^m)\frac{\partial\phi}{\partial\nu}dS(x)dt\right| &\le \frac{2T}{Nw_NR^{N-1}}\int_{\partial B_R}u(x)dS(x)\\ &= \frac{2T}{Nw_N}⨍_{\partial B_R}u(x)ds(x)\\ &\to 0 \text{ as } R\to\infty. \end{aligned}$$

From where, letting $R\to\infty$ in **(2.1.15)** and using monotone convergence theorem, we find:

$$\int_0^T\Big(z_1^m(0,t) - w_\infty^m(0,t)\Big)dt + \int_{\mathbb{R}^N}\rho(x)\Gamma(x)\Big(z_1(x,T) - w_\infty(x,T)\Big)dx = 0,$$

from where using that $\Gamma,\rho>0$ and since $T>0$ is arbitrary we obtain $w_\infty(x,t) = z_1(x,t)$ for all $x\in\mathbb{R}^N$ and for all $t>0$.

On the other hand, since $u,v\in L^\infty(\mathbb{R}^N)$ and they are positive functions, there exists $t_R>0$ such that

$$u^{\frac{1}{m}} < \frac{v^{\frac{1}{m}}}{t_R^{\frac{1}{m-1}}} \text{ in } B_R.$$

Thus, from comparison result, we have

$$w_R(x,t) \leq \frac{Cv^{\frac{1}{m}}}{(t+t_R)^{\frac{1}{m-1}}} \text{ in } B_R \times (0,T),$$

and thus

$$w_R(x,t) \leq \frac{Cv^{\frac{1}{m}}}{t^{\frac{1}{m-1}}} \text{ in } B_R \times (0,T).$$

Finally passing to the limit, as $R \to \infty$, in the above inequality and considering that $T > 0$ is arbitrary, we obtain

$$w_\infty(x,t) \leq \frac{Cv^{\frac{1}{m}}}{t^{\frac{1}{m-1}}} \text{ in } \mathbb{R}^N \times (0,\infty),$$

that is to say:

$$\frac{Cu^{\frac{1}{m}}}{(t+1)^{\frac{1}{m-1}}} \leq \frac{Cv^{\frac{1}{m}}}{t^{\frac{1}{m-1}}},$$

which implies that $u \leq v$ as $t \to \infty$. Therefore, $u = v$.

□

Remark 2.1.5. There exist other bounded positive solutions of **(2.1.1)** which do not satisfy **(2.1.8)**. In fact, given any positive constant a, there exists a solution of **(2.1.10)** satisfying

$$\liminf_{|x|\to\infty} u(x) = a.$$

Indeed, consider the problem

$$\begin{cases} -\boldsymbol{\Delta u = \rho(x)u^{\alpha}} & \text{in} \quad \boldsymbol{B_R} \\ \boldsymbol{u = a} & \text{on} \quad \boldsymbol{\partial B_R.} \end{cases} \tag{2.1.16}$$

It is clear that $\underline{u} = a$ is a lower solution of Problem **(2.1.16)**. Moreover, there exists $C > 0$ large enough such that

$$\overline{u} = \frac{C}{|x|^{N-2}} * \rho + a$$

is an upper solution of **(2.1.16)** for all $R > 0$. Therefore using lower and upper solution technique, as in **Theorem (2.1.1)**, we find $u_\infty \in H^1\big(\mathbb{R}^N\big) \cap L^\infty\big(\mathbb{R}^N\big)$ solution of:

$$-\Delta u = \rho(x)u^{\alpha} \quad \text{in } \mathbb{R}^N,$$

satisfying

$$a \leq u(x) \leq \frac{C}{|x|^{N-2}} * \rho + a \ \text{ in } \mathbb{R}^N,$$

hence

$$\liminf_{|x|\to\infty} u(x) = a.$$

2.2 The linear Schrödinger equation

After having investigated the existence and uniqueness of solutions of the equation:

$$-\Delta u = \rho(x)u^{\alpha} \quad \text{in } \mathbb{R}^N,$$

which was obtained assuming the property (H), that is, there is a bounded solution of linear equation:

$$-\Delta u = \rho(x) \text{ in } \mathbb{R}^N$$

a natural generalization of this problem is to study existence of bounded solutions for the following linear Schrödinger equation

$$-\Delta u + V(x)u = \rho(x) \text{ in } \mathbb{R}^N \tag{2.2.1}$$

and later a nonlinear Schrödinger equation of type

$$-\Delta u + V(x)u = f(x,u) \text{ in } \mathbb{R}^N. \tag{NS}$$

Recently J. Cardoso, P. Cerda, D. Pereira and P. Ubilla (see [11]) completely develop of the problem **(2.2.1)** in order to obtain the existence of at least two solutions of Problem **(NS)** where the models $f(x,u)$ studied were:

i) $\rho(x)u^q$ ii) $\lambda\rho(x)(u+1)^p$ iii) $\lambda\rho(x)(u^q+u^p)$,

where $0 < q < 1 < p < 2^* - 1$ and ρ satisfies property (H), in all three cases.

However, this section will give the main results regarding linear Schrödinger **(2.2.1)**, which will be used in **Chapter 3**.

As noted in [11] $u \in C^2(\mathbb{R}^N)$ given by

$$u(x) = \frac{1}{(1+|x|^{\beta})^{\gamma}},$$

where $\beta > 2$ and $\gamma \geq 0$ is a classical solution of the linear schrödinger equation **(2.2.1)** for V and ρ given by

$$V(x) = \frac{\gamma(\gamma+1)\beta^2|x|^{2(\beta-1)}}{(1+|x|^{\beta})^2} \quad \text{and} \quad \rho(x) = \frac{\gamma(\beta+N-2)|x|^{\beta-2)}}{(1+|x|^{\beta})^{\gamma+1}},$$

while the linear Schrödinger equation **(2.2.1)** with

$$V(x) = \frac{1}{1+|x|^{\alpha}} \quad \text{and} \quad \rho(x) = \frac{1}{1+|x|^{\beta}}$$

does not have any bounded solution for any $\alpha > \beta$ and $\beta \in (0,2]$ (see **Example 2.2.9**). This tells us that the existence and nonexistence of bounded positive solutions for equation **(2.2.1)** is related to the growth of V and ρ. For this reason, we will assume:

1. $V : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous potential and there exist $a, A, \alpha > 0$, such that

$$\frac{a}{1+|x|^{\alpha}} \leq V(x) \leq \frac{A}{1+|x|^{\alpha}} \quad \text{for all} \quad x \in \mathbb{R}^N. \tag{H_V^{α}}$$

2. $\rho : \mathbb{R}^N \to \mathbb{R}$ is smooth and there exists $k_1 > 0$, such that

$$0 < \rho(x) \le \frac{k_1}{1 + |x|^\beta} \quad \text{for all} \quad x \in \mathbb{R}^N, \qquad (H_\rho)$$

with $\alpha + \beta > 4$.

Next we present the theorems related to Problem **(2.2.1)**. For this purpose we introduce a compatibility condition between ρ and V, given in [11].

Definition 2.2.1. Suppose that ρ has the property (H) and let U be the bounded solution of $(\mathbf{P_e})$ i.e.

$$-\Delta U = \rho(x) \quad \text{in} \quad \mathbb{R}^N.$$

We say that V and ρ are compatible if

$$\frac{1}{|x|^{N-2}} * (VU) \in L^\infty(\mathbb{R}^N).$$

Remark 2.2.1. Notice that V and ρ are compatible says that the product VU also has the property (H).

Lemma 2.2.2. *Assume that ρ satisfies (H_ρ) and V satisfies (H_V^α) with $\alpha \in (0, 2)$. Then V and ρ are compatible*

Proof. Since ρ satisfies (H_ρ), by **Proposition 2.1.14** there exists $C > 0$ such that

$$U(x) \le \frac{C}{1 + |x|^{\beta-2}} \quad \text{for all} \quad x \in \mathbb{R}^N.$$

Thus from (H_V^2) we also have

$$V(x)U(x) \le \frac{AC}{1 + |x|^{\alpha+\beta-2}} \quad \text{for all} \quad x \in \mathbb{R}^N,$$

which implies

$$\frac{1}{|x|^{N-2}} * \big(V(x)U(x)\big) \in L^\infty(\mathbb{R}^N)$$

whenever $\alpha + \beta > 4$. □

Theorem 2.2.3. *If V and ρ are compatible, then the linear Schrödinger equation (2.2.1) has a bounded positive solution.*

Proof. Let U be the only one bounded positive solution of $(\mathbf{P_e})$, vanishing in infinity, given by **Lemma 2.1.6**, and let u_R be a nonnegative solution of the problem

$$\begin{cases} -\Delta u_R + V(x)u_R = \rho(x) & \text{in} \quad B_R \\ u_R = 0 & \text{on} \quad \partial B_R. \end{cases} \qquad \mathbf{(2.2.2)}$$

Since $V \in L^\infty(B_R)$ and $\rho(x) > 0$ it follows that $u_R \ge 0$ in B_R and $u_R \ne 0$. Note that

$$-\Delta u_R \le -\Delta u_R + V(x)u_R = \rho(x) = -\Delta U \quad \text{in} \quad B_R.$$

Using the maximum principle, we see that $u_R(x) \leq U(x)$ in B_R. Moreover, u_R is increasing in R, that is, if $R' > R$ then $u_{R'} \geq u_R$ in B_R. In fact, $u_{R'}$ is an upper solution of (2.2.2) in B_R then

$$-\Delta(u_{R'} - u_R) + V(x)(u_{R'} - u_R) \geq 0 \text{ in } B_R$$

with $u_{R'} - u_R = u_{R'} \geq 0$ on ∂B_R. The maximum principle implies that $u_{R'} \geq u_R$ in B_R. Now define $v_R = U - u_R$. Then, v_R is solution of the equation

$$\begin{cases} \boldsymbol{-\Delta v_R = V(x) u_R} & \text{in} \quad \boldsymbol{B_R} \\ \boldsymbol{u_R = U} & \text{on} \quad \boldsymbol{\partial B_R.} \end{cases}$$

Furthermore $v_R \leq U$ for all $R > 0$ and $v_R \geq v_{R'}$ for $R \leq R'$. Using the Green's representation formula, we see that

$$v_R(x) = c_1 \int_{\partial B_R} \frac{R^2 - |x^2|}{R|x-y|^N} U(y)dy + c_2 \int_{B_R} G_R(x,y)V(y)u_R(y)dy.$$

Let $U_V := \lim\limits_{R\to\infty} u_R$. Using monotone convergence, we obtain

$$\lim_{R\to\infty} \int_{B_R} G_R(x,y)V(y)u_R(y)dy = c \int_{\mathbb{R}^N} \frac{V(y)U_V(y)}{|x-y|^{N-2}} dy.$$

On the other hand, since $|x-y| \geq |y| - |x| = R - |x|$ for any $|y| = R$, it follows that

$$\frac{1}{|x-y|^N} \leq \frac{1}{(R-|x|)^N},$$

for large values of R, which implies

$$\begin{aligned} \int_{\partial B_R} \frac{R^2 - |x^2|}{R|x-y|^N} U(y)dy &\leq \frac{1}{R^{N-1}} \int_{\partial B_R} \frac{(R^2 - |x^2|)R^{N-2}}{(R-|x|)^N} U(y)dy \\ &\leq \frac{(R^2 - |x^2|)R^{N-2}}{(R-|x|)^N} \fint_{\partial B_R} U(y)dy \to 0, \quad R \to \infty. \end{aligned}$$

Therefore $v := \lim\limits_{R\to\infty} v_R$ is given by

$$v(x) = c \int_{\mathbb{R}^N} \frac{V(y)U_V(y)}{|x-y|^{N-2}} dy.$$

Using that $U_V \leq U$ and the compatibility between V and ρ, we get $v \in L^\infty(\mathbb{R}^N)$. Moreover the function v satisfies

$$-\Delta v = V(x)U_V(x) \quad \text{in} \quad \mathbb{R}^N$$

and

$$v = \lim_{R\to\infty} v_R = \lim_{R\to\infty} (U - u_R) = U - U_V.$$

Thus we obtain

$$\rho(x) + \Delta U_V = -\Delta(U - U_V) = -\Delta v = V(x)U_V \quad \text{in} \quad \mathbb{R}^N$$

or equivalently

$$-\Delta U_V + V(x)U_V = \rho(x) \quad \text{in} \quad \mathbb{R}^N.$$

Moreover we see that

$$\lim_{|x|\to\infty} U_V = \lim_{|x|\to\infty} (U - v) = 0.$$

□

As an application of the previous theorem, we can build many examples of linear Schrödinger equation **(2.2.1)** which have at least one bounded positive solution.

Example 2.2.4. Let $\alpha > \gamma \geq 0$ and $\beta > 2$ with $\alpha + \beta > 4 + \gamma$, then the problem

$$-\Delta u = \frac{|x|^\gamma}{1+|x|^\alpha} u = \frac{1}{1+|x|^\beta} \quad \text{in} \quad \mathbb{R}^N$$

has a unique bounded positive solution satisfying

$$U_V(x) = U(x) - c\int_{\mathbb{R}^N} \frac{U_V(y)|y|^\gamma}{(1+|y|^\alpha)|x-y|^{N-2}} dy$$

where U is the only bounded positive solution of $(\mathbf{P_e})$.

Next, we will assume two new hypotheses, in order to obtain a lower bound for the solution of the problem **(2.2.1)**.

Lemma 2.2.5. *Assume that ρ satisfies (H_ρ) and*

$$\frac{k_0}{1+|x|^\beta} \leq \rho(x) \text{ for all } x \in \mathbb{R}^N \tag{H'_ρ}$$

for some constant $k_0 > 0$ with $\beta > 2$ and also assume $V : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous potential verifying

$$\frac{c_1\mu}{1+|x|^2} \leq V(x) \leq \frac{\mu}{1+|x|^2} \text{ for all } x \in \mathbb{R}^N \tag{H_V^2}$$

for some $0 < c_1 < 1$.
*Then the bounded positive solution U_V, for the linear schrödinger equation **(2.2.1)**, satisfies*

$$(1 - c_2\mu)U(x) \leq U_V(x) \leq U(x) \text{ for all } x \in \mathbb{R}^N.$$

for some constant $c_2 > 0$.

Proof. Hypotheses (H_ρ), (H_V^2) imply that V and ρ are compatible for any $\mu > 0$. From the proof of the **Theorem 2.2.3**, we see that the solution U_V of **(2.2.1)** satisfies

$$U_V(x) = U(x) - c\int_{\mathbb{R}^N} \frac{V(y)U_V(y)}{|x-y|^{N-2}} dy$$

or equivalently

$$\frac{U_V(x)}{U(x)} = 1 - \frac{c}{U(x)}\int_{\mathbb{R}^N} \frac{V(y)U_V(y)}{|x-y|^{N-2}} dy.$$

The hypothesis (H_ρ) implies that there exists $c_2 > 0$ such that

$$\frac{U(x)}{1+|x|^2} \leq \frac{c_2}{1+|x|^\beta} \quad \text{in} \quad \mathbb{R}^N.$$

Then we have

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{V(y)U_V(y)}{|x-y|^{N-2}}dy &\leq \int_{\mathbb{R}^N} \frac{\mu U(y)}{1+|y|^2}.\frac{1}{|x-y|^{N-2}}dy \\ &\leq c_2\mu \int_{\mathbb{R}^N} \frac{1}{(1+|y|^\beta)|x-y|^{N-2}} \\ &= c_1\mu U_0(x)\end{aligned}$$

where U_0 is the unique bounded positive solution, vanishing in infinity, of

$$-\Delta U_0 = \frac{1}{1+|x|^\beta} \text{ in } \mathbb{R}^N.$$

Since (H'_ρ) we see that $k_0 U_0 \leq U$, then

$$\frac{c}{U(x)} \int_{\mathbb{R}^N} \frac{V(y)U_V(y)}{|x-y|^{N-2}}dy \leq c_3\mu$$

for some constant $c_3 > 0$. Therefore

$$\frac{U_V(x)}{U(x)} \geq (1 - c_4)$$

for some constant $c_4 > 0$. □

Corollary 2.2.6. *Assume the hypotheses (H_ρ), (H'_ρ) and (H^2_V). Let U_V be the bounded positive solution for the linear Schrödinger equation **(2.2.1)**. Then there exists $C_\mu > 0$ such that*

$$\frac{\rho(x)}{V(x)U_V(x)} \leq C_\mu \quad \text{for all} \quad x \in \mathbb{R}^N.$$

Proof. By **Lemma 2.2.5**, there exist $0 < c_1 < 1$ and $c_2 > 0$ such that

$$\begin{aligned}\limsup_{|x|\to\infty} \frac{\rho(x)}{V(x)U_V(x)} &\leq \limsup_{|x|\to\infty} \frac{\rho(x)}{(1-c_2\mu)V(x)U(x)} \\ &\leq \limsup_{|x|\to\infty} \frac{\rho(x)(1+|x|^2)}{c_1\mu(1-c_2\mu)U(x)} \\ &\leq \limsup_{|x|\to\infty} \frac{k_1(1+|x|^2)}{c_1\mu(1-c_2\mu)(1+|x|^\beta)U(x)} \\ &\leq C_\mu \limsup_{|x|\to\infty} \frac{(1+|x|^2)|x|^{\beta-2}}{1+|x|^\beta} \\ &< C_\mu < \infty\end{aligned}$$

for some constant $C_\mu > 0$.

□

As a final comment on equation **(2.2.1)**, we will give two results, if you are interested, you can see their proofs in [11], which show nonexistence of bounded solutions to the problem **(2.2.1)**.

Theorem 2.2.7. *Let $\rho \in L^\infty(\mathbb{R}^N)$ be a positive potential such that does not satisfies property (H) and V satisfying*

$$\lim_{|x|\to\infty} \frac{V(x)}{\rho(x)} = 0.$$

Then, the linear Schrödinger equation ***(2.2.1)*** *does not have positive and bounded solutions.*

The previous theorem is a consequence of the following theorem that generalizes the **Lemma 2.1.6**

Lemma 2.2.8. *Assume that $\rho \in L^\infty_{loc}(\mathbb{R}^N)$ is a nonnegative only outside of some ball centered at the origin, i.e. there exists a constant $M > 0$ such that*

$$\rho(x) \geq 0 \text{ a.e. in } |x| \geq M,$$

and that ρ is not identically zero, then the equation

$$-\Delta u = \rho(x) \text{ in } \mathbb{R}^N$$

has a bounded solution iff

$$\frac{1}{|x|^{N-2}} * \rho \in L^\infty(\mathbb{R}^N).$$

Example 2.2.9. As an application of **Theorem 2.2.7** , if $\beta < \alpha$ and $\beta \geq 2$, then the linear Schrödinger equation **(2.2.1)**

$$-\Delta u = \frac{1}{1+|x|^\alpha} u = \frac{1}{1+|x|^\beta} \quad \text{in} \quad \mathbb{R}^N,$$

has no bounded positive solution.

Chapter 3

Elliptic systems involving Schrödinger operators

In this Chapter, assuming the conditions (H_V^α), (H_ρ) and using the upper and lower solutions techniques, we first prove the existence of a bounded positive solution of System:

$$\begin{cases} -\Delta u + V_1(x)u = \lambda\rho_1(x)(u+1)^r(v+1)^p & \text{in} \quad \mathbb{R}^N \\ -\Delta v + V_2(x)v = \mu\rho_2(x)(u+1)^q(v+1)^s & \text{in} \quad \mathbb{R}^N, \\ u(x), v(x) \to 0 & \text{as} \quad |x| \to \infty. \end{cases} \tag{$S_{\lambda,\mu}$}$$

where $\lambda, \mu > 0$, $p, q, r, s \geq 0$, $N \geq 3$. Furthermore, by imposing some restrictions on the powers p, q, r, s without additional hypotheses on the weights ρ_i, we obtain a second solution using variational methods. In this context we consider two particular cases: a gradient system and a Hamiltonian system.

3.1 Existence and nonexistence results. General case

The proof of existence of a solution of System $(S_{\lambda,\mu})$ follows the line of [9], [11] and [35], that is to say, we will apply some monotonicity methods. Since we are working with systems, we will use the lower and upper solutions technique developed by Montenegro [35] to obtain a solution (u_R, v_R) of

$$\begin{cases} -\Delta u + V_1(x)u = \lambda\rho_1(x)(u+1)^r(v+1)^p & \text{in} \quad B_R \\ -\Delta v + V_2(x)v = \mu\rho_2(x)(u+1)^q(v+1)^s & \text{in} \quad B_R \\ u = 0 = v & \text{on} \quad \partial B_R \end{cases} \tag{$S_{R,\lambda,\mu}$}$$

where $B_R = \{x \in \mathbb{R}^N : \ |x| \leq R\}$. Then, we will prove that (u_R, v_R) is an increasing sequence of bounded solutions which converge to a bounded solution of $(S_{\lambda,\mu})$, when the radius R tends to infinity.

The proof of **Theorem 1** is based on the following Lemma.

Lemma 3.1.1. *Assume that $p, q, r, s \geq 0$. Let U_{V_i} be a bounded positive solution of*

$$\begin{cases} -\Delta u + V_i(x)u = \rho_i(x) & \text{in} \quad \mathbb{R}^N \\ u(x) \to 0 & \text{as} \quad |x| \to \infty. \end{cases} \tag{3.1.1}$$

Then there is $\Lambda > 0$, which does not depend on R, such that if $0 < \lambda, \mu < \Lambda$, the System $(\mathbf{S}_{\mathbf{R},\lambda,\mu})$ has a minimal positive solution (u_R, v_R), which is increasing with R and satisfies

$$u_R \leq U_{V_1} \text{ and } v_R \leq U_{V_2}. \tag{3.1.2}$$

Proof. Let $R > 0$. Notice first that $(\underline{u}, \underline{v}) = (0, 0)$ is a lower solution of $(\mathbf{S}_{\mathbf{R},\lambda,\mu})$ for any $\lambda, \mu \in (0, \infty)$. To construct an upper solution, we define $(\overline{u}, \overline{v}) = (U_{V_1}, U_{V_2})$. Then, $(\overline{u}, \overline{v})$ is an upper solution of $(\mathbf{S}_{\mathbf{R},\lambda,\mu})$ if and only if

$$\begin{cases} -\Delta(U_{V_1}) + V_1(x)(U_{V_1}) \geq \lambda\rho_1(x)\big(U_{V_1} + 1\big)^r\big(U_{V_2} + 1\big)^p \\ -\Delta(U_{V_2}) + V_2(x)(U_{V_2}) \geq \mu\rho_2(x)\big(U_{V_1} + 1\big)^q\big(U_{V_2} + 1\big)^s. \end{cases}$$

These two inequalities hold if

$$\begin{cases} 1 \geq \lambda\big(\|U_{V_1}\|_\infty + 1\big)^r\big(\|U_{V_2}\|_\infty + 1\big)^p \\ 1 \geq \mu\big(\|U_{V_1}\|_\infty + 1\big)^q\big(\|U_{V_2}\|_\infty + 1\big)^s. \end{cases}$$

Thus, we see that there exists $\Lambda > 0$ such that for $0 < \lambda, \mu \leq \Lambda$, the pair $(\overline{u}, \overline{v})$ is an upper solution of $(\mathbf{S}_{\mathbf{R},\lambda,\mu})$, for any $R > 0$. Therefore, from **Lemma 1.6.2**, there is a solution $(\overline{u}_R, \overline{v}_R)$ of $(\mathbf{S}_{\mathbf{R},\lambda,\mu})$ satisfying

$$0 \leq \overline{u}_R \leq U_{V_1} \quad \text{and} \quad 0 \leq \overline{v}_R \leq U_{V_2}.$$

Furthermore, we have $\overline{u}_R \neq 0$ and $\overline{v}_R \neq 0$ in B_R, then by maximum principle

$$0 < \overline{u}_R \leq U_{V_1} \quad \text{and} \quad 0 < \overline{v}_R \leq U_{V_2}.$$

Now we will show existence of minimal solution for every $R > 0$. In fact, let (z, w) be any bounded positive solution of $(\mathbf{S}_{\mathbf{R},\lambda,\mu})$, which we already know exists, and let (u_0, v_0) be a bounded positive solution of

$$\begin{cases} -\Delta u + V_1(x)u = \lambda\rho_1(x) & \text{in} \quad B_R \\ -\Delta v + V_2(x)v = \mu\rho_2(x) & \text{in} \quad B_R \\ u = 0 = v & \text{on} \quad \partial B_R. \end{cases}$$

Then there is a sequence (u_n, v_n) of solutions for

$$\begin{cases} -\Delta u_n + V_1(x)u_n = \lambda\rho_1(x)(u_{n-1} + 1)^r(v_{n-1} + 1)^p & \text{in} \quad B_R \\ -\Delta v_n + V_2(x)v_n = \mu\rho_2(x)(u_{n-1} + 1)^q(v_{n-1} + 1)^s & \text{in} \quad B_R \\ u_n = 0 = v_n & \text{on} \quad \partial B_R \end{cases}$$

for every $n \in \mathbb{N}$. We see that

$$\begin{aligned} -\Delta(z - u_0) + V_1(x)(z - u_0) &= \lambda\rho_1(x)\Big((z + 1)^r(w + 1)^p - 1\Big) \\ &\geq 0 \quad \text{in} \quad B_R. \end{aligned}$$

Therefore, by maximum principle, $u_0 \leq z$ in B_R. In the same way we see that $v_0 \leq w$ in B_R. Now we claim that

$$u_n \leq z \quad \text{and} \quad v_n \leq w \quad \text{in} \quad B_R \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.1.3}$$

Indeed, by principle of induction, let $k \in \mathbb{N}$ and suppose **(3.1.3)** holds for $n = k$. Then

$$-\Delta(z - u_{k+1}) + V_1(x)(z - u_{k+1}) = \lambda\rho_1(x)\Big((z+1)^r(w+1)^p - (u_k+1)^r(v_k+1)^p\Big)$$
$$\geq 0 \quad \text{in} \quad B_R.$$

Thus, by maximum principle we have $u_{k+1} \leq z$. Similarly $v_{k+1} \leq w$. Then, the principle of induction is satisfied. Furthermore, and again, using maximum principle, we have

$$u_0 \leq u_1 \leq \ldots \leq u_n \quad \text{and} \quad v_0 \leq v_1 \leq \ldots \leq v_n.$$

Therefore, there exist the limits

$$\lim_{n\to\infty} u_n(x) := u_R(x), \quad \lim_{n\to\infty} v_n(x) := v_R(x) \quad \text{for every} \quad x \in B_R$$

and thus (u_R, v_R) is a positive solution of $(\mathbf{S}_{\mathbf{R},\lambda,\mu})$ satisfying

$$u_R \leq z \quad \text{and} \quad v_R \leq w \quad \text{in} \quad B_R.$$

Since (z, w) is an arbitrary solution of $(\mathbf{S}_{\mathbf{R},\lambda,\mu})$ we have that (u_R, v_R) is a minimal positive solution of $(\mathbf{S}_{\mathbf{R},\lambda,\mu})$.
We claim that (u_R, v_R) is increasing with R, that is, if $R' > R > 0$, then

$$u_R \leq u_{R'} \quad \text{and} \quad v_R \leq v_{R'} \quad \text{in} \quad B_R.$$

Indeed, if $R' > R$ then $(u_{R'}, v_{R'})$ is an upper solution of $(\mathbf{S}_{\mathbf{R},\lambda,\mu})$ and $(\underline{u}, \underline{v}) = (0,0)$ is a lower solution of $(\mathbf{S}_{\mathbf{R},\lambda,\mu})$, thus, there exists a solution $(\overline{u}, \overline{v})$ of $(\mathbf{S}_{\mathbf{R},\lambda,\mu})$, such that

$$0 \leq \overline{u} \leq u_{R'} \quad \text{and} \quad 0 \leq \overline{v} \leq v_{R'} \quad \text{in} \quad B_R.$$

Since (u_R, v_R) is the minimal solution of $(\mathbf{S}_{\mathbf{R},\lambda,\mu})$, we have

$$u_R \leq \overline{u} \leq u_{R'} \quad \text{and} \quad v_R \leq \overline{v} \leq v_{R'} \quad \text{in} \quad B_R.$$

This shows that (u_R, v_R) is increasing with R. Again, using the minimality of (u_R, v_R), we obtain

$$u_R \leq U_{V_1} \quad \text{and} \quad v_R \leq U_{V_2}.$$

□

Proof of Theorem 1. Let $0 < \lambda, \mu < \Lambda$, $R > 0$ and (u_R, v_R) be the increasing sequence of solution of $(\mathbf{S}_{\mathbf{R},\lambda,\mu})$, given by **Lemma 3.1.1**. Thus, there exist the limits

$$\lim_{R\to\infty} u_R(x) := u(x) \quad \text{and} \quad \lim_{R\to\infty} v_R(x) := v(x) \quad \text{for every} \quad x \in \mathbb{R}^N.$$

We claim that (u, v) is a bounded positive solution of $(\mathbf{S}_{\lambda,\mu})$. Indeed, since ρ_1 satisfies property (H_ρ) and $u, v \in L^\infty(\mathbb{R}^N)$ there exists $U_1 \in L^\infty(\mathbb{R}^N)$ satisfying

$$-\Delta U_1 = \lambda\rho_1(x)(u+1)^r(v+1)^p \text{ in } \mathbb{R}^N, \tag{3.1.4}$$

and

$$\lim_{|x|\to\infty} U_1(x) = 0. \tag{3.1.5}$$

Let U_{1R} denote the solution of equation

$$\begin{cases} -\boldsymbol{\Delta U = \lambda\rho_1(x)(u_R+1)^r(v_R+1)^p} & \text{in} \quad \boldsymbol{B_R} \\ \boldsymbol{U = 0} & \text{on} \quad \boldsymbol{\partial B_R.} \end{cases}$$

It is clear that U_{1R} is positive in B_R. Now, we will show that it is bounded, and increasing with R. Indeed, since u_R and v_R are bounded from above by u and v, respectively, we have

$$\begin{aligned} -\Delta(U_{1R} - U_1) &= \lambda\rho_1(x)\Big((u_R+1)^r(v_R+1)^p - (u+1)^r(v+1)^p\Big) \\ &\le 0 \text{ in } B_R. \end{aligned}$$

Likewise, for $R' > R$, since u_R, v_R are increasing with R, we see that

$$\begin{aligned} -\Delta(U_{1R} - U_{1R'}) &= \lambda\rho_1(x)\Big((u_R+1)^r(v_R+1)^p - (u_{R'}+1)^r(v_{R'}+1)^p\Big) \\ &\le 0 \text{ in } B_R. \end{aligned}$$

Therefore, using the maximum principle, we see that U_{1R} is increasing and

$$U_{1R}(x) \le U_1(x) \quad \text{in} \quad B_R \tag{3.1.6}$$

for all $R > 0$. Using Green's identities, we see that

$$U_{1R}(x) = c\int_{B_R} \lambda G_R(x,y)\rho(y)(u_R(y)+1)^r(v_R(y)+1)^p dy, \;\; x \in B_{R.}$$

Since $\overline{U}_1 := \lim_{R\to\infty} U_{1R}$ is bounded by U_1 in $\mathbb{R}^N$, thus by monotone convergence we have the following representation formula

$$\overline{U}_1(x) = c\int_{\mathbb{R}^N} \frac{\lambda\rho(y)(u(y)+1)^r(v(y)+1)^p}{|x-y|^{N-2}} dy,$$

and therefore the function $\overline{U}_1$ provides a bounded solution of (3.1.4) (see **Lemma 2.1.6**). Moreover by (3.1.5) and (3.1.6), $\overline{U}_1$ satisfies

$$\lim_{|x|\to\infty} \overline{U}_1(x) = 0,$$

and uniqueness of solution of (3.1.4)-(3.1.5) (see **Lemma 2.1.10**) implies that

$$\lim_{R\to\infty} U_{1R} = \overline{U}_1 = U_1.$$

Also note that

$$-\Delta u_R \le -\Delta u_R + V_1(x)u_R = \lambda\rho_1(x)(u_R+1)^r(v_R+1)^p = -\Delta U_{1R} \;\; \text{in} \;\; B_R.$$

Using the maximum principle, we see that $u_R(x) \le U_{1R}(x)$ in B_R. We define $w_{1R} = U_{1R} - u_R$. Then w_{1R} is a solution of the equation

$$\begin{cases} \boldsymbol{-\Delta w_{1R} = V_1(x)u_R} & \text{in} \quad \boldsymbol{B_R} \\ \boldsymbol{w_{1R} = 0} & \text{on} \quad \boldsymbol{\partial B_R.} \end{cases}$$

Also, w_{1R} is increasing with R and $w_{1R} \leq U_1$ for all $R > 0$. Using Green's representation formula, we see that

$$w_{1R}(x) = c\int_{B_R} G_R(x,y)V_1(y)u_R(y)dy, \ x \in B_{R.}$$

Using monotone convergence, we obtain

$$\lim_{R\to\infty}\int_{B_R} G_R(x,y)V_1(y)u_R(y)dy = c\int_{\mathbb{R}^N}\frac{V_1(y)u(y)}{|x-y|^{N-2}}dy.$$

Therefore, $w_1 := \lim_{R\to\infty} w_{1R}$ is given by

$$w_1(x) = c\int_{\mathbb{R}^N}\frac{V_1(y)u(y)}{|x-y|^{N-2}}dy.$$

Using that $u \leq U_{V_1}$ and the compatibility between V_1 and ρ_1, we get $w_1 \in L^\infty(\mathbb{R}^N)$. Note that the function w_1 satisfies

$$-\Delta w_1 = V_1(x)u(x) \ \text{ in } \ \mathbb{R}^N$$

and

$$w_1 = \lim_{R\to\infty} w_{1R} = \lim_{R\to\infty}(U_{1R} - u_R) = U_1 - u \ \ (\text{ See } \textbf{Lemma 2.1.6}).$$

Thus, we obtain

$$-\Delta(U_1 - u) = -\Delta w_1 = V_1(x)u \ \text{ in } \ \mathbb{R}^N$$

or equivalently,

$$-\Delta u + V_1(x)u = -\Delta U_1 = \lambda\rho_1(x)(u+1)^r(v+1)^p \ \text{ in } \ \mathbb{R}^N,$$

that is,

$$-\Delta u + V_1(x)u = \lambda\rho_1(x)(u+1)^r(v+1)^p \ \text{ in } \ \mathbb{R}^N.$$

Proceeding in the same way with v, and since (u_R, v_R) satisfy **(3.1.2)**, we conclude that (u, v) is a bounded positive solution of the system

$$\begin{cases} -\Delta u + V_1(x)u = \lambda\rho_1(x)(u+1)^r(v+1)^p & \text{in} \quad \mathbb{R}^N \\ -\Delta v + V_2(x)v = \mu\rho_2(x)(u+1)^q(v+1)^s & \text{in} \quad \mathbb{R}^N, \\ u(x), v(x) \to 0 & \text{as} \quad |x| \to \infty. \end{cases}$$

□

Remark 3.1.1. We notice that is not difficult to verify that (u, v) is a minimal solution of $(\mathbf{S}_{\lambda,\mu})$.

Next, we will give the proof of **Theorem 2** which is the converse of **Theorem 1**.

Proof of Theorem 2. Let (u,v) be a bounded positive solution of system $(\mathbf{S}_{\lambda,\mu})$. We will follow the line developed in [35, Theorem 5.2], however the computations here are more delicate. In fact, we consider the auxiliary function $w=(u+1)^a(v+1)^b$, with $a=1-r$ and $b=1-s$. The following claims hold:

(1) $0<a,b<1$.

(2) $\nabla w = a(u+1)^{a-1}(v+1)^b\nabla u + b(u+1)^a(v+1)^{b-1}\nabla v$.

(3) $$|\nabla w|^2 = a^2(u+1)^{2(a-1)}(v+1)^{2b}|\nabla u|^2 + b^2(u+1)^{2a}(v+1)^{2(b-1)}|\nabla v|^2 + 2ab(u+1)^{2a-1}(v+1)^{2b-1}\langle\nabla u,\nabla v\rangle.$$

(4) $$\frac{|\nabla w|^2}{w} = a^2(u+1)^{a-2}(v+1)^b|\nabla u|^2 + b^2(u+1)^a(v+1)^{b-2}|\nabla v|^2 + 2ab(u+1)^{a-1}(v+1)^{b-1}\langle\nabla u,\nabla v\rangle.$$

(5) $$\Delta w = a(u+1)^{a-1}(v+1)^b\Delta u + 2ab(u+1)^{a-1}(v+1)^{b-1}\langle\nabla u,\nabla v\rangle + b(u+1)^a(v+1)^{b-1}\Delta v + a(a-1)(u+1)^{a-2}(v+1)^b|\nabla u|^2 + b(b-1)(u+1)^a(v+1)^{b-2}|\nabla v|^2.$$

(6) $2ab(u+1)^{a-1}(v+1)^{b-1}\langle\nabla u,\nabla v\rangle \le a^2(u+1)^{a-2}(v+1)^b|\nabla u|^2 + b^2(u+1)^a(v+1)^{b-2}|\nabla v|^2$.

Combining (4) and (6), it follows that

$$2ab(u+1)^{a-1}(v+1)^{b-1}\langle\nabla u,\nabla v\rangle \le \frac{|\nabla w|^2}{2w}.$$

So, from (1), (5) and since (u,v) is solution of $(\mathbf{S}_{\lambda,\mu})$, we obtain

$$\begin{aligned}
\Delta w &\le \frac{|\nabla w|^2}{2w} + a(u+1)^{a-1}(v+1)^b\Delta u + b(u+1)^a(v+1)^{b-1}\Delta v\\
\Delta w &\le \frac{|\nabla w|^2}{2w} + a(u+1)^{a-1}(v+1)^b\Big(-\lambda\rho_1(x)(u+1)^r(v+1)^p + V(x)u\Big)\\
&\quad + b(u+1)^a(v+1)^{b-1}\Big(-\mu\rho_2(x)(u+1)^q(v+1)^s + V(x)v\Big)\\
&\le \frac{|\nabla w|^2}{2w} - a\lambda\rho_1(x)(v+1)^{b+p} - b\mu\rho_2(x)(u+1)^{a+q} + (a+b)V(x)(u+1)^a(v+1)^b.
\end{aligned}$$

On the other hand, let $\eta\in\mathbb{R}$ satisfying

$$\frac{1}{\eta} = \frac{1}{\frac{b+p}{b}} + \frac{1}{\frac{a+q}{a}}.$$

Since $pq<(r-1)(s-1)$ it follows that $1/2\le\eta<1$. Using Young's inequality, we have

$$\Big((a\lambda\rho_1(x))^{\frac{b\eta}{b+p}}(v+1)^{b\eta}\Big)\Big((b\mu\rho_2(x))^{\frac{a\eta}{a+q}}(u+1)^{a\eta}\Big) \le \left(\frac{a\eta}{a+q}\right)a\lambda\rho_1(x)(v+1)^{b+p} + \left(\frac{b\eta}{b+p}\right)b\mu\rho_2(x)(u+1)^{a+q},$$

which implies that

$$\begin{aligned}\Delta w + (a\lambda\rho_1(x))^{\frac{a\eta}{a+q}}(b\mu\rho_2(x))^{\frac{b\eta}{b+p}}w^\eta &\le \frac{|\nabla w|^2}{2w} + (a+b)V(x)w \\ &+ \left(\frac{a\eta}{a+q} - 1\right)a\lambda\rho_1(x)(v+1)^{b+p} + \left(\frac{b\eta}{b+p} - 1\right)b\mu\rho_2(x)(u+1)^{a+q} \\ &\le \frac{|\nabla w|^2}{2w} + (a+b)V(x)w.\end{aligned}$$

Defining the function $z = \dfrac{1}{1-\eta}w^{1-\eta}$, we get

$$\begin{aligned}-\Delta z &= \eta w^{-\eta-1}|\nabla w|^2 - w^{-\eta}\Delta w \\ &\ge \eta w^{-\eta-1}|\nabla w|^2 - \frac{1}{2}w^{-\eta-1}|\nabla w|^2 + (a\lambda\rho_1(x))^{\frac{a\eta}{a+q}}(b\mu\rho_2(x))^{\frac{b\eta}{b+p}} - (a+b)V(x)w^{1-\eta} \\ &\ge (a\lambda\rho_1(x))^{\frac{a\eta}{a+q}}(b\mu\rho_2(x))^{\frac{b\eta}{b+p}} - (1-\eta)(a+b)V(x)z.\end{aligned}$$

Since $b_1\rho(x) \le \rho_2(x)$, $0 < (1-\eta)(a+b) < 1$ and V be a nonnegative potential, we obtain

$$-\Delta z + V(x)z \ge c_1\rho_1(x),$$

where $c_1 = (a\lambda)^{\frac{a\eta}{a+q}}(b_1 b\mu)^{\frac{b\eta}{b+p}}$. Therefore, we conclude that $c_1^{-1}z$ is a bounded positive upper solution of

$$-\Delta u + V(x)u = \rho_1(x) \text{ in } \mathbb{R}^N$$

satisfying

$$\lim_{|x|\to\infty} z(x) = 0. \qquad \textbf{(3.1.7)}$$

For every $R > 0$ denote by u_R the increasing positive solution of the problem

$$\begin{cases} -\Delta u_R + V(x)u_R = \rho_1(x) & \text{in} \quad B_R \\ u_R = 0 & \text{on} \quad \partial B_R. \end{cases}$$

By maximum principle, we have

$$u_R \le c_1^{-1}z \quad \text{in} \quad B_R.$$

So $U_V(x) := \lim_{R\to\infty} u_R(x)$ there exists for every x in $\mathbb{R}^N$ and satisfies

$$U_V \le c_1^{-1}z \text{ in } \mathbb{R}^N. \qquad \textbf{(3.1.8)}$$

By **(3.1.7)** and **(3.1.7)** we have U_V is a bounded positive solution of

$$-\Delta U_V + V(x)U_V = \rho_1(x) \quad \text{in} \quad \mathbb{R}^N$$

satisfying

$$\lim_{|x|\to\infty} U_V(x) = 0.$$

In the same way it is shown that the linear schrödinger equation

$$-\Delta u + V(x)u = \rho_2(x) \quad \text{in} \quad \mathbb{R}^N$$

has a bounded positive solution. □

Remark 3.1.2. Note that the use of the auxiliary functions $w = u^a v^{1-a}$ have already been used for different purposes; for instance, see [7, 34, 37, 39].

Next, we will prove a nonexistence result.

Lemma 3.1.2. *Assume that $r, s > 1$ and $p, q \geq 0$. Then there exists $\overline{\Lambda} > 0$ such that for all $\lambda, \mu > \overline{\Lambda}$, System $(\mathbf{S}_{\lambda,\mu})$ has no bounded positive solutions.*

Proof. Let (z, w) a bounded positive solution of System $(\mathbf{S}_{\lambda,\mu})$. Let φ_1, ψ_1 be the positive eigenfunction associated to the first eigenvalue $\overline{\lambda}_1$, $\overline{\mu}_1$ of the Schrödinger equation

$$\begin{cases} -\Delta\varphi_1 + V_1(x)\varphi_1 = \overline{\lambda}_1\rho_1(x)\varphi_1 & \text{in} \quad B_R \\ -\Delta\psi_1 + V_2(x)\psi_1 = \overline{\mu}_1\rho_2(x)\psi_1 & \text{in} \quad B_R \\ \varphi_1 = 0 = \psi_1 & \text{on} \quad \partial B_R. \end{cases}$$

By Hopf's Lemma

$$\frac{\partial\varphi_1}{\partial\nu} < 0 \quad \text{on} \quad B_R.$$

Extending φ_1 by 0 in $\mathbb{R}^N \setminus B_R$, and using Green's formulas we have

$$\begin{aligned} \int_{\mathbb{R}^N} \nabla z \nabla\varphi_1 dx &= \int_{\partial B_R} z\frac{\partial\varphi_1}{\partial\nu} dS + \int_{B_R} z(-\Delta\varphi_1)dx \\ &\leq \int_{B_R} z(-\Delta\varphi_1)dx \\ &= \overline{\lambda}_1 \int_{B_R} \rho_1(x)z\varphi_1 dx - \int_{B_R} V_1(x)z\varphi_1 dx. \end{aligned}$$

Therefore

$$\int_{\mathbb{R}^N} \big(\nabla z\nabla\varphi_1 + V_1(x)z\varphi_1\big)dx \leq \overline{\lambda}_1 \int_{B_R} \rho_1(x)z\varphi_1 dx. \tag{3.1.9}$$

On the other hand, using that $p \geq 0$, $r > 1$ and that (z, w) is a solution of $(\mathbf{S}_{\lambda,\mu})$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} \big(\nabla z\nabla\varphi_1 + V_1(x)z\varphi_{11}\big)dx &= \lambda \int_{\mathbb{R}^N} \rho_1(x)\big(z+1\big)^r\big(w+1\big)^p\varphi_1 dx \\ &\geq \lambda \int_{B_R} \rho_1(x)z\varphi_1 dx \end{aligned}$$

Thus, by (3.1.9), it follows that

$$\lambda \int_{B_R} \rho_1(x)z\varphi_1 dx \leq \overline{\lambda}_1 \int_{B_R} \rho_1(x)z\varphi_1 dx.$$

Since z, ρ_1, $\varphi_1 > 0$ it implies that $\lambda \leq \overline{\lambda}_1$. Likewise $\mu \leq \overline{\mu}_1$. Taking $\overline{\Lambda} = \max\{\overline{\lambda}_1, \overline{\mu}_1\}$ the proof is finished. ∎

Before proving **Theorem 3** let us to introduce the parameter λ^* given by:

$$\lambda^* = \sup\{\lambda > 0 : \exists\mu > 0, (\mathbf{S}_{\lambda,\mu}) \text{ has a bounded positive solution}\},$$

which is well defined and finite by **Theorem 1** and **Lemma 3.1.2**.

Lemma 3.1.3. *Assume that $r, s > 1$ and $p, q \geq 0$. Then for all $0 < \lambda < \lambda^*$, there is $\mu > 0$ such that System ($\mathbf{S}_{\lambda,\mu}$) has a bounded positive solution.*

Proof. Fixed $\lambda \in (0, \lambda^*)$, there exists $\lambda_0 \in (\lambda, \lambda^*)$ such that System ($\mathrm{S}_{\lambda_0,\mu}$) has a solution (u_{λ_0}, v_μ), for some $\mu > 0$. It is clear that (u_{λ_0}, v_μ) is an upper solution to the System ($\mathbf{S}_{\mathbf{R},\lambda,\mu}$) for all $R > 0$ and $(0, 0)$ is a lower solution to these systems. Then from **Lemma 1.6.2**, there is a bounded positive solution of ($\mathbf{S}_{\mathbf{R},\lambda,\mu}$). This implies that System ($\mathbf{S}_{\mathbf{R},\lambda,\mu}$) has a minimal positive solution $(z_{R,\lambda}, w_{R,\mu})$. Thus, by a similar argument to that in the proof of **Theorem 1** there exists (z_λ, w_μ) a minimal bounded positive solution of System ($\mathbf{S}_{\lambda,\mu}$) given by

$$z_\lambda(x) = \lim_{R\to\infty} z_{R,\lambda}(x), \quad w_\mu(x) = \lim_{R\to\infty} w_{R,\mu}(x), \quad x \in \mathbb{R}^N.$$

□

Proof of Theorem 3. We define the function $\Gamma : (0, \lambda^*) \to [0, \infty)$ by

$$\Gamma(\lambda) = \sup\{\mu > 0 : (\mathbf{S}_{\lambda,\mu}) \text{ has a bounded positive solution}\}.$$

We claim that Γ it is a nonincreasing function. Indeed, let $0 < \lambda \leq \lambda_0 < \lambda^*$. By **Lemma 3.1.3** there is $\mu_0 > 0$ such that System ($\mathrm{S}_{\lambda_0,\mu_0}$) has a solution $(u_{\lambda_0}, v_{\mu_0})$. Thus $\Gamma(\lambda_0) \geq \mu_0$. We claim that $\Gamma(\lambda) \geq \Gamma(\lambda_0)$. In fact, if $\Gamma(\lambda) < \Gamma(\lambda_0)$, then we could find $\mu_1 \in (\Gamma(\lambda), \Gamma(\lambda_0))$ and from the definition of $\Gamma(\lambda_0)$ there would be $\mu_2 \in (\mu_1, \Gamma(\lambda_0))$ and $(z_{\lambda_0}, w_{\mu_2})$ solution of ($\mathrm{S}_{\lambda_0,\mu_2}$), which is an upper solution to the System ($\mathrm{S}_{\mathrm{R}\lambda,\mu_2}$) for all $R > 0$ and therefore there is a bounded positive solution of ($\mathrm{S}_{\lambda,\mu_2}$). This implies that $\mu_2 \leq \Gamma(\lambda)$, which is a contradiction. Also since $\Gamma(0, \lambda^*)$ is an interval we conclude that Γ is continuous.

It is clear that System ($\mathbf{S}_{\lambda,\mu}$) has at least one bounded positive solution if $0 < \mu < \Gamma(\lambda)$ and has no solution if $\mu > \Gamma(\lambda)$, for every $\lambda \in (0, \lambda^*)$. □

Remark 3.1.3. In order to obtain the existence of a minimal bounded positive solution of System ($\mathbf{S}_{\lambda,\mu}$), we only assume that the powers are positive. Now, to define the curve Γ, it was necessary to assume $r, s > 1$ and $p, q \geq 0$. Note that when the powers satisfy $r, s \geq 0$, $p, q > 1$, we may obtain a similar result to that of **Theorem 3**.

3.2 The gradient system.

This section is devoted to the proof of **Theorem 4**. Note that under the conditions (H_ρ) and (H_V^α) with $\alpha \in (0,2]$ and $\alpha+\beta>4$, the potentials ρ and V are compatible, thus using a similar argument as in **Theorem 1**, **Lemma 3.1.2** and **Lemma 3.1.3**, we have that there exists $\lambda^*>0$ such that for every $0<\lambda<\lambda^*$ there is a bounded positive solution of System (3.2), which we will denote by $(u_{1,\lambda}, v_{1,\lambda})$, while for $r,s>1$ and $\lambda>\lambda^*$ there are no bounded positive solutions, and so, **Theorem 4** part i) is proved.

Before proving the existence of the second solution of System $(\mathbf{GS}_\lambda)$, let us observe that the *most natural energy functional* $\mathfrak{J}_\lambda : E \to \mathbb{R}$, associated to the gradient system $(\mathbf{GS}_\lambda)$ should be given by

$$\mathfrak{J}_\lambda(u,v) = \frac{1}{2}\|(u,v)\|^2 - \lambda \int_{\mathbb{R}^N} \rho(x) F(u,v)dx,$$

where $F:\mathbb{R}^2 \to \mathbb{R}$ is defined by

$$F(u,v) = (u+1)^{r+1}(v+1)^{s+1},$$

where we have assumed that $r,s>1$ and $r+s<2^*-2$. However, it is not well defined because the Sobolev embeddings do not work. This is mainly due to the behavior near zero of the nonlinearities and the fact that the $\rho(x)$ coefficient does not necessarily satisfy any integrability hypothesis. For this reason, in order to show the existence of a second solution for System $(\mathbf{GS}_\lambda)$, we will consider the following auxiliary system

$$\begin{cases} -\Delta u + V(x)u = \lambda\rho(x)f(x,u,v) & \text{in} \quad \mathbb{R}^N \\ -\Delta v + V(x)v = \lambda\rho(x)g(x,u,v) & \text{in} \quad \mathbb{R}^N \end{cases} \qquad (\mathbf{GS}_\mathbf{A}^\lambda)$$

where the functions f, g are defined by

$$f(x,u,v) = f_1\big(u_{1,\lambda}+u^+, v_{1,\lambda}+v^+\big) - f_1\big(u_{1,\lambda}, v_{1,\lambda}\big)$$

and

$$g(x,u,v) = f_2\big(u_{1,\lambda}+u^+, v_{1,\lambda}+v^+\big) - f_2\big(u_{1,\lambda}, v_{1,\lambda}\big),$$

where for simplicity we have denoted $u_{1,\lambda}, v_{1,\lambda}$ instead of $u_{1,\lambda}(x), v_{1,\lambda}(x)$, and where

$$f_1(u,v) = \frac{\partial F}{\partial u} \quad \text{and} \quad f_2(u,v) = \frac{\partial F}{\partial v}.$$

Now we define $G:\mathbb{R}^{N+2}\to\mathbb{R}$ by

$$G(x,u,v) = F\big(u_{1,\lambda}+u^+, v_{1,\lambda}+v^+\big) - F\big(u_{1,\lambda}, v_{1,\lambda}\big) - \Big(f_1\big(u_{1,\lambda}, v_{1,\lambda}\big)u^+ + f_2\big(u_{1,\lambda}, v_{1,\lambda}\big)v^+\Big).$$

Then

$$\frac{\partial G}{\partial u} = f(x,u,v) \quad \text{and} \quad \frac{\partial G}{\partial v} = g(x,u,v).$$

This shows that the auxiliary problem $(\mathbf{GS}_\mathbf{A}^\lambda)$ is also a gradient system. Clearly, if (u,v) is a solution for the auxiliary system $(\mathbf{GS}_\mathbf{A}^\lambda)$, then $(u_{1,\lambda}+u, v_{1,\lambda}+v)$ is a solution of System (3.2).

Note that this type of idea has already been used in [11] (in $\mathbb{R}^N$) for the scalar equation. In what follows, we will prove that the energy functional associated to the auxiliary system **($\mathbf{GS}^\lambda_\mathbf{A}$)** given by

$$J_\lambda(u,v) = \frac{1}{2}\|(u,v)\|^2 - \lambda \int_{\mathbb{R}^N} \rho(x) G(x,u,v)dx$$

unlike the *most natural energy functional*, is well defined on E. In addition, J_λ belongs to the $C^1(E,\mathbb{R})$ space and has a critical point at the Mountain Pass level for $\lambda > 0$ sufficiently small.

Lemma 3.2.1. *The functional J_λ associated to **($GS^\lambda_\mathbf{A}$)** is well defined in E.*

Proof. First notice that for $a, c > 0$ and $b, d \geq 0$, defining $h : [0,1] \to \mathbb{R}$ by

$$h(\eta) = (a + \eta b)^t (c + \eta d)^l$$

the Mean Value Theorem gives us the existence of $\xi \in (0,1)$ such that

$$\begin{aligned}(a+b)^t(c+d)^l - a^t c^l &= h(1) - h(0) = h'(\xi) \\ &= t(a+\xi b)^{t-1}(c+\xi d)^l b + l(a+\xi b)^t (c+\xi d)^{l-1} d.\end{aligned}$$

So, it is not difficult to see that the following inequalities hold

$$(I_{tl}) \qquad (a+b)^t(c+d)^l - a^t c^l \leq \begin{cases} t(a+b)^{t-1}(c+d)^l b + l(a+b)^t(c+d)^{l-1} d & \text{if } t,l \geq 1 \\ t a^{t-1}(c+d)^l b + l(a+b)^t(c+d)^{l-1} d & \text{if } 0 \leq t < 1, l \geq 1 \\ t(a+b)^{t-1}(c+d)^l b + l(a+b)^t c^{l-1} d & \text{if } t \geq 1, 0 \leq l < 1 \\ t a^{t-1}(c+d)^l b + l(a+b)^t c^{l-1} d & \text{if } 0 < t,l < 1. \end{cases}$$

Also, note that the last inequality is valid for $0 \leq t < 1, 0 < l < 1$ or $0 < t < 1, 0 \leq l < 1$. From now on, for simplicity, let us denote $z_\lambda := u_{1,\lambda} + u + 1$ and $w_\lambda := v_{1,\lambda} + v + 1$. Using these inequalities, we have

$$z_\lambda^{r+1} w_\lambda^{s+1} - \left(u_{1,\lambda}+1\right)^{r+1}\left(v_{1,\lambda}+1\right)^{s+1} \leq (r+1) z_\lambda^r w_\lambda^{s+1} u + (s+1) z_\lambda^{r+1} w_\lambda^s v.$$

Again, using inequality (I_{rs}), we see that

$$(r+1)\Big(z_\lambda^r w_\lambda^{s+1} - \left(u_{1,\lambda}+1\right)^r \left(v_{1,\lambda}+1\right)^{s+1}\Big) u \leq (r+1)\Big(r z_\lambda^{r-1} w_\lambda^{s+1} u + (s+1) z_\lambda^r w_\lambda^s v\Big) u.$$

Similarly we get

$$(s+1)\Big(z_\lambda^{r+1} w_\lambda^s - \left(u_{1,\lambda}+1\right)^{r+1}\left(v_{1,\lambda}+1\right)^s\Big) v \leq (s+1)\Big(r z_\lambda^r w_\lambda^s u + (s+1) z_\lambda^{r+1} w_\lambda^{s-1} v\Big) v.$$

Since $u_{1,\lambda},\ v_{1,\lambda}$ are bounded, from definition of G we see that there is $C_1 > 0$ such that

$$G(x,u,v) \leq C_1(u^2 + v^2) \quad \text{for} \quad u + v \approx 0.$$

It is also clear that there is $C_2 > 0$ such that

$$G(x,u,v) \leq C_2(u+v)^{r+s+2} \quad \text{for} \quad u + v \approx \infty.$$

Thus, there exists $C > 0$ such that

$$G(x,u,v) \leq C\left(u^2 + v^2 + (u+v)^{r+s+2}\right) \quad \text{for all } x \in \mathbb{R}^N \text{ and } u,v \geq 0. \qquad \textbf{(3.2.2)}$$

Then using **Proposition 1.2.3** we see that the functional associated to $(\mathbf{GS_A^\lambda})$ given by is well defined. Moreover $J_\lambda \in C^1\big(E, \mathbb{R}\big)$ with

$$J'_\lambda(u,v)(\psi,\phi) = \langle (u,v),(\psi,\varphi)\rangle - \lambda \int_{\mathbb{R}^N} \rho(x)\big(f(x,u,v)\psi + g(x,u,v)\phi\big)dx,$$

for any $(u,v),(\psi,\phi) \in E$. □

The nonlinearity G satisfies the following property which is more general than the classical Ambrosetti-Rabinowitz condition:

Lemma 3.2.2. *There exist $\theta \in \big(2, 2^*\big)$ and $C > 0$ such that*

$$uf(x,u,v) + vg(x,u,v) - \theta G(x,u,v) \geq -C\big(u^2 + v^2\big)$$

for all $x \in \mathbb{R}^N$ and $u, v > 0$.

The proof of the lemma above is a direct consequence of the following two lemmas .

Lemma 3.2.3. *There exist $2 < \theta < 2^*$ and $r_0 > 0$ such that*

$$0 < \theta G(x,u,v) \leq uf(x,u,v) + vg(x,u,v) \tag{3.2.3}$$

for all $x \in \mathbb{R}^N$ and every $u, v > 0$ such that $u + v \geq r_0$.

Proof. Let $u, v \geq 0$ and define $h : [0,\infty) \to \mathbb{R}$ by

$$h(t) = \big(u_{1,\lambda} + tu + 1\big)^{r+1}\big(v_{1,\lambda} + tv + 1\big)^{s+1}.$$

Then, by mean value theorem, there exist $\xi \in (0,1)$ such that $h(1) - h(0) = h'(\xi)$. Thus

$$\begin{aligned}
&F\big(u_{1,\lambda} + u, v_{1,\lambda} + v\big) - F\big(u_{1,\lambda}, v_{1,\lambda}\big) = h(1) - h(0) = h'(\xi)\\
&= (r+1)\Big(u_{1,\lambda} + \xi u + 1\Big)^r \Big(v_{1,\lambda} + \xi v + 1\Big)^{s+1} u + (s+1)\Big(u_{1,\lambda} + \xi u + 1\Big)^{r+1}\Big(v_{1,\lambda} + \xi v + 1\Big)^s v\\
&> (r+1)\Big(u_{1,\lambda} + 1\Big)^r \Big(v_{1,\lambda} + 1\Big)^{s+1} u + (s+1)\Big(u_{1,\lambda} + 1\Big)^{r+1}\Big(v_{1,\lambda} + 1\Big)^s v.
\end{aligned}$$

It follows that $G(x,u,v) > 0$ for all $x \in \mathbb{R}^N$ and every $u, v \geq 0$.

On the other hand, since we are looking for $\theta > 2$, and since $f_1\big(u_{1,\lambda}, v_{1,\lambda}\big)u + f_2\big(u_{1,\lambda}, v_{1,\lambda}\big)v > 0$ to prove **(3.2.3)**, it is sufficient to show

$$\frac{f_1\big(u_{1,\lambda} + u, v_{1,\lambda} + v\big)u + f_2\big(u_{1,\lambda} + u, v_{1,\lambda} + v\big)v}{F\big(u_{1,\lambda} + u, v_{1,\lambda} + v\big) - F\big(u_{1,\lambda}, v_{1,\lambda}\big)} \geq \theta. \tag{3.2.4}$$

for all $x \in \mathbb{R}^N$ and every $u, v > 0$ such that $u + v \geq r_0$. For this purpose, we need to verify that there exists $\theta \in \big(2, 2^*\big)$ such that

$$\liminf_{u+v \to +\infty} h(x,u,v) \geq \theta. \tag{3.2.5}$$

where

$$h(x,u,v) := \frac{(r+1)u}{z_\lambda} + \frac{(s+1)v}{w_\lambda},$$

with $z_\lambda = u_{1,\lambda} + u + 1$ and $w_\lambda = v_{1,\lambda} + v + 1$. Indeed, by (3.1.5) there exists a constant $c > 0$ such that the functions $u_{1,\lambda}, v_{1,\lambda}$ are bounded by c. Then we have

$$\begin{aligned} h(x,u,v) &\geq \min\{r+1, s+1\} \left(\frac{u}{z_\lambda} + \frac{v}{w_\lambda} \right) \\ &\geq \min\{r+1, s+1\} \left(\frac{u}{u+v+\bar{c}} + \frac{v}{u+v+\bar{c}} \right) \\ &= \min\{r+1, s+1\} \left(\frac{u+v}{u+v+\bar{c}} \right) \\ &\to \min\{r+1, s+1\}, \quad \text{as} \quad u+v \to \infty, \end{aligned}$$

where $\bar{c} = c+1$. Therefore, there is $\theta \in \big(2, \min\{r+1, s+1\}\big)$ verifying (3.2.5). So for $u, v > 0$ and $x \in \mathbb{R}^N$, we have

$$\begin{aligned} \frac{f_1(u_{1,\lambda}+u, v_{1,\lambda}+v)u + f_2(u_{1,\lambda}+u, v_{1,\lambda}+v)v}{F(u_{1,\lambda}+u, v_{1,\lambda}+v) - F(u_{1,\lambda}, v_{1,\lambda})} &= \frac{(r+1)z_\lambda^r w_\lambda^{s+1} u + (s+1) z_\lambda^{r+1} w_\lambda^s v}{z_\lambda^{r+1} w_\lambda^{s+1} - \big(u_{1,\lambda}+1\big)^{r+1} \big(v_{1,\lambda}+1\big)^{s+1}} \\ &\geq \frac{(r+1)z_\lambda^r w_\lambda^{s+1} u + (s+1) z_\lambda^{r+1} w_\lambda^s v}{z_\lambda^{r+1} w_\lambda^{s+1}} \\ &= \frac{(r+1)u}{z_\lambda} + \frac{(s+1)v}{w_\lambda} \\ &= h(x,u,v) \geq \theta \quad \text{for} \quad u+v \approx \infty. \end{aligned}$$

This concludes the proof. □

Lemma 3.2.4. *Let $\theta \in (2, 2^*)$. Then, there exists $r_1 > 0$ small enough and $C > 0$ such that*

$$uf(x,u,v) + vg(x,u,v) - \theta G(x,u,v) \geq -C\big(u^2 + v^2\big)$$

for all $x \in \mathbb{R}^N$ and every $u, v > 0$ such that $u + v \leq r_1$.

Proof. Let $x \in \mathbb{R}^N$, $u, v > 0$ and z_λ, w_λ as in the previous lemma. Note that

$$\begin{aligned} F(u_{1,\lambda}+u, v_{1,\lambda}+v) - F\big(u_{1,\lambda}, v_{1,\lambda}\big) &= z_\lambda^{r+1} w_\lambda^{s+1} - \big(u_{1,\lambda}+1\big)^{r+1} \big(v_{1,\lambda}+1\big)^{s+1} \\ &\leq (r+1) z_\lambda^r w_\lambda^{s+1} u + (s+1) w_\lambda^s \big(u_{1,\lambda}+1\big)^{r+1} v. \end{aligned}$$

This implies that there exist $C > 0$ and $r_1 > 0$ small enough such that

$$\begin{aligned} uf(x,u,v) + vg(x,u,v) - \theta G(x,u,v) &\geq -\theta G(x,u,v) \\ &\geq -C(u^2+v^2), \end{aligned}$$

for all $x \in \mathbb{R}^N$ and $u + v \leq r_1$. □

The following lemma is a simple consequence of the definition of G.

Lemma 3.2.5. *Let $\theta \in \big(2, 2^*\big)$. Then, there exists $r_2 > 0$ such that*

$$G(x,u,u) \geq u^\theta, \ \textit{for all } x \in \mathbb{R}^N \textit{ and } u \geq r_2.$$

The next lemma says that J_λ has the Mountain Pass geometry.

Lemma 3.2.6.

i) There exist $\lambda_1^* > 0$ and $r_0, \ a > 0$ such that

$$J_\lambda(u,v) \geq a \textit{ if } \|(u,v)\| = r_0 \textit{ for every } \ \lambda \in (0, \lambda_1^*).$$

ii) There exists $(u,v) \in E$ with

$$\|(u,v)\| > r_0 \textit{ and } J_\lambda(u,v) < 0.$$

Proof.

i) By **Proposition 1.2.3** and **(3.2.2)**, there exists $C > 0$ such that

$$\begin{aligned} J_\lambda(u,v) &= \frac{1}{2}\|(u,v)\|^2 - \lambda \int_{\mathbb{R}^N} \rho(x) G(x,u,v)dx \\ &\geq \frac{1}{2}\|(u,v)\|^2 - \lambda C\big(\|(u,v)\|^2 + \|(u,v)\|^{r+s+2}\big). \end{aligned}$$

Then there exists $0 < \lambda_1^* < \lambda^*$ such that for every $0 < \lambda < \lambda_1^*$ we have that if $\|(u,v)\| = \lambda$, then

$$J_\lambda(u,v) \geq \lambda^2 \left(\frac{1}{2} - 2\lambda C\right) := a_\lambda > 0.$$

ii) This is a consequence of **Lemma 3.2.5**.

□

Lemma 3.2.7. *There exists $\lambda_2^* > 0$ enough small such that the functional J_λ satisfies the Palais-Smale condition for every $\lambda \in (0, \lambda_2^*)$.*

Proof. Let $(u_n, v_n) \subset E$ be a Palais-Smale sequence at level c. By **Lemma 3.2.2** and **Proposition 1.2.3**, for n sufficiently large, we have

$$\begin{aligned} & c + \frac{o(1)}{\theta}\|(u_n,v_n)\| = J_\lambda(u_n,v_n) - \frac{1}{\theta} J'_\lambda(u_n,v_n)(u_n,v_n) \\ &= \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|(u_n,v_n)\|^2 + \frac{\lambda}{\theta}\int_{\mathbb{R}^N} \rho(x)\Big(f(x,u_n,v_n)u_n + g(x,u_n,v_n)v_n - \theta G(x,u_n,v_n)\Big)dx \\ &\geq \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|(u_n,v_n)\|^2 - \frac{\lambda C}{\theta}\int_{\mathbb{R}^N} \rho(x)\big(u_n^2 + v_n^2\big)dx \\ &\geq \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|(u_n,v_n)\|^2 - \frac{\lambda C}{\theta}\Big(\|u_n\|^2_{H^1_V(\mathbb{R}^N)} + \|v_n\|^2_{H^1_V(\mathbb{R}^N)}\Big) \\ &= \left(\frac{1}{2} - \frac{1}{\theta} - \frac{\lambda C}{\theta}\right)\|(u_n,v_n)\|^2. \end{aligned}$$

Since $\theta \in \big(2, 2^*\big)$, there exists $0 < \lambda_2^* < \lambda_1^*$ such that for every $0 < \lambda < \lambda_2^*$ we have that

$$\left(\frac{1}{2} - \frac{1}{\theta} - \frac{\lambda C}{\theta}\right) > 0,$$

which implies that the sequence (u_n, v_n) is bounded in E. Then, through standard argument and **Proposition 1.2.3**, there is a subsequence still denoted by (u_n, v_n) that converges in E. □

Finally, By **Lemma 3.2.1**, **Lemma 3.2.6** and **Lemma 3.2.7** there exists $0 < \lambda^{**} \leq \lambda^*$ such that the functional J_λ is well defined and satisfies the conditions of the Mountain Pass Theorem for every $\lambda \in (0, \lambda^{**})$. Therefore, there exists a (u, v) solution of $(\mathbf{GS_A^\lambda})$ for any $\lambda \in (0, \lambda^{**})$, which allows us to conclude the proof of **Theorem 4** part ii).

3.3 The Hamiltonian system

Theorem 1 gives us the existence of $\Lambda > 0$ such that Hamiltonian System **(HS$_\lambda$)** has at least one bounded positive solution for every $0 < \lambda < \Lambda$.
On the other hand, if (u, v) were a bounded positive solution of **(HS$_\lambda$)**, then we would have

$$\int_{\mathbb{R}^N} \big(\nabla u \nabla \varphi + V(x)u\varphi\big)dx \geq \lambda \int_{B_R} \rho(x)v\varphi dx$$

and

$$\int_{\mathbb{R}^N} \big(\nabla v \nabla \varphi + V(x)v\varphi\big)dx \geq \lambda \int_{B_R} \rho(x)u\varphi dx,$$

for all $\varphi \in H^1_V(\mathbb{R}^N)$. Let φ_1 the positive eigenfunction associated to the first eigenvalue $\overline{\lambda}_1$ of the Schrödinger equation

$$\begin{cases} -\Delta\varphi_1 + V(x)\varphi_1 = \overline{\lambda}_1\rho(x)\varphi_1 & \text{in} \quad B_R \\ \varphi_1 = 0 & \text{on} \quad \partial B_R. \end{cases}$$

In a similar way as in **(3.1.7)**, we have

$$\int_{\mathbb{R}^N} \big(\nabla u \nabla \varphi_1 + V(x)u\varphi_1\big)dx \leq \overline{\lambda}_1 \int_{B_R} \rho(x)u\varphi_1 dx$$

and

$$\int_{\mathbb{R}^N} \big(\nabla v \nabla \varphi_1 + V(x)v\varphi_1\big)dx \leq \overline{\lambda}_1 \int_{B_R} \rho(x)v\varphi_1 dx,$$

These four inequalities would imply that

$$\lambda \int_{B_R} \rho(x)\big(u+v\big)\varphi_1 dx \leq \overline{\lambda}_1 \int_{B_R} \rho(x)\big(u+v\big)\varphi_1 dx.$$

Since $u, v, \rho > 0$ and $\varphi_1 > 0$ we have $\lambda \leq \overline{\lambda}_1$. Therefore using an argument similar to the **Lemma 3.1.3** the proof of **Theorem 5** part i) is complete.

Now, let $R > 0$ be, by choosing $\gamma > q$ such that $p\gamma < 1$ is possible to find $M > 1$ large enough such that

$$\begin{cases} M \geq \lambda(M^\gamma \|U_{V_2}\|_\infty + 1)^p \\ M^\gamma \geq \mu(M\|U_{V_1}\|_\infty + 1)^q, \end{cases}$$

where U_{V_1}, U_{V_2} is a bounded positive solution of **(3.1.1)**. Thus, the couple $(MU_{V_1}, M^\gamma U_{V_2})$ is an upper solution of **($S_{R,\lambda,\mu}$)** for every $\lambda, \mu > 0$, and since $(\underline{u}, \underline{v}) = (0, 0)$ is a lower solution of **($S_{R,\lambda,\mu}$)**, by virtue **Lemma 1.6.2** and, following the argument in **Lemma 3.1.1** and **Theorem 1**, we obtain existence of at least one bounded positive solution of Hamiltonian System **(HS$_\lambda$)** for all $\lambda > 0$, which proves **Theorem 5** part ii).

Finally, in order to prove **Theorem 5** part iii), without loss of generality we will assume that $p > 1$ and let $(u_{1,\lambda}, v_{1,\lambda})$ be a bounded positive solution of **(HS$_\lambda$)** given by **Theorem 5** i). In a similar way as in a gradient system, to show the existence of a second solution for the System **(HS$_\lambda$)** we will show the existence of at least one solution for the following auxiliary system.

$$\begin{cases} -\Delta u + V(x)u = \lambda\rho(x)f(x,v) & \text{in} \quad \mathbb{R}^N \\ -\Delta v + V(x)v = \lambda\rho(x)g(x,u) & \text{in} \quad \mathbb{R}^N, \end{cases} \qquad (\mathrm{HS}_{\mathrm{A}}^{\lambda})$$

with

$$f(x,v) := h_1(v_{1,\lambda}+v^+) - h_1(v_{1,\lambda}), \quad g(x,u) := h_2(u_{1,\lambda}+u^+) - h_2(u_{1,\lambda})$$

and

$$h_1(v) = \frac{\partial \mathcal{H}}{\partial v}, \quad h_2(u) = \frac{\partial \mathcal{H}}{\partial u},$$

where $\mathcal{H} : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$\mathcal{H}(u,v) = \frac{(u+1)^{q+1}}{q+1} + \frac{(v+1)^{p+1}}{p+1}.$$

Define $H : \mathbb{R}^{N+2} \to \mathbb{R}$ by

$$H(x,u,v) = \mathcal{H}(u_{1,\lambda}+u^+, v_{1,\lambda}+v^+) - \mathcal{H}(u_{1,\lambda}, v_{1,\lambda}) - \big(h_1(v_{1,\lambda})v^+ + h_2(u_{1,\lambda})u^+\big).$$

Then

$$\frac{\partial H}{\partial v} = f(x,v) \ \text{ and } \ \frac{\partial H}{\partial u} = g(x,u).$$

This shows that the auxiliary problem $(\mathrm{HS}_{\mathrm{A}}^{\lambda})$ is also a Hamiltonian system.

To show the existence of a nontrivial solution of the auxiliary problem $(\mathrm{HS}_{\mathrm{A}}^{\lambda})$, we will use the technique developed in [42] (see also [32]), in which the authors show the existence of at least one positive solution for a Hamiltonian system of the form:

$$\begin{cases} -\Delta u + V(x)u = \rho_1(x)f(v) & \text{in} \quad \mathbb{R}^N \\ -\Delta v + V(x)v = \rho_2(x)g(u) & \text{in} \quad \mathbb{R}^N, \end{cases}$$

Since the nonlinearities of our system $(\mathrm{HS}_{\mathrm{A}}^{\lambda})$ are not of separate variables, we cannot directly use their argument. However, by taking λ small enough, we can adapt their argument for our case. In this line, we will use the linking result due to Li and Szulkin [31].

Now, we will prove that the energy functional associated to the auxiliary system $(\mathrm{HS}_{\mathrm{A}}^{\lambda})$ given by

$$I_\lambda(u,v) = \int_{\mathbb{R}^N} \Big(\nabla u \nabla v + V(x)uv\Big)dx - \lambda \int_{\mathbb{R}^N} \rho(x)H(x,u,v)dx$$

is well defined on E. In fact, the following result holds.

Lemma 3.3.1. *The functional I_λ associated to $(\mathrm{HS}_{\mathrm{A}}^{\lambda})$ is well defined in E.*

Proof. Let $u, v \geq 0$. The inequality (I_{tl}) in the **Lemma 3.2.1** proof, with $l = 0$ tells us:

$$\big(a+b\big)^t - a^t \leq \begin{cases} t(a+b)^{t-1}b & \text{for all} \quad b \geq 0, \ a > 0 \text{ if } t \geq 1 \\ ta^{t-1}b & \text{for all} \quad b \geq 0, \ a > 0 \text{ if } 0 < t < 1. \end{cases}$$

Using this inequality twice, for $0 < q \leq 1$, we see that

$$\begin{aligned} \frac{z_\lambda^{q+1} - (u_{1,\lambda}+1)^{q+1}}{q+1} - (u_{1,\lambda}+1)^q u &\leq \big(z_\lambda^q - (u_{1,\lambda}+1)^q\big)u \\ &\leq q(v_{1,\lambda}+\xi u+1)^{q-1}u^2. \\ &\leq u^2, \ \text{ for all } u \geq 0. \end{aligned}$$

Similarly, if $q > 1$, we have

$$\begin{aligned}\frac{z_\lambda^{q+1} - (u_{1,\lambda}+1)^{q+1}}{q+1} - (u_{1,\lambda}+1)^q u &\le \big(z_\lambda^q - (u_{1,\lambda}+1)^q\big)u \\ &\le q z_\lambda^{q-1} u^2.\end{aligned}$$

Thus, using the fact that $v_{1,\lambda}$ is bounded, we see that there exists $C_1 > 0$ such that

$$\frac{z_\lambda^{q+1} - (u_{1,\lambda}+1)^{q+1}}{q+1} - (u_{1,\lambda}+1)^q u \le \begin{cases} u^2 & \text{for all} \quad u \ge 0 \text{ and } 0 < q \le 1 \\ C_1 u^2 & \text{for} \quad u \approx 0 \text{ and } q > 1. \end{cases}$$

In the similar way, since $u_{1,\lambda}$ is bounded, there exists $C_2 > 0$ such that

$$\frac{w_\lambda^{p+1} - (v_{1,\lambda}+1)^{p+1}}{p+1} - (v_{1,\lambda}+1)^p v \le C_2(v^2 + v^{p+1}), \text{ for all } v \ge 0,$$

where $z_\lambda = u_{1,\lambda} + u + 1$ and $w_\lambda = v_{1,\lambda} + v + 1$. Thus, from definition of H we obtain the existence of $C > 0$ such that

$$H(x,u,v) \le \begin{cases} C(u^2 + v^2 + v^{p+1}) & \text{if} \quad 0 < q \le 1 \\ C(u^2 + v^2 + u^{q+1} + v^{p+1}) & \text{if} \quad q > 1, \end{cases} \tag{3.3.2}$$

for all $x \in \mathbb{R}^N$ and $u, v \ge 0$. Since $p > 1$ and $p, q < 2^* - 1$, by **Proposition 1.2.3**, the functional I_λ is well defined and $I_\lambda \in C^1\big(E, \mathbb{R}\big)$ with Fréchet derivate given by

$$I'_\lambda(u,v)(\psi,\phi) = \langle (u,v),(\phi,\psi)\rangle - \lambda \int_{\mathbb{R}^N} \rho(x)\Big(f(x,v)\phi + g(x,u)\psi\Big)dx$$

for any $(u,v),(\psi,\phi) \in E$. □

The next result says that f and g are superlinear at infinity.

Lemma 3.3.2. *If $q > 1$, then the following limits hold*

$$\lim_{v\to\infty} \frac{f(x,v)}{v} = \infty = \lim_{u\to\infty} \frac{g(x,u)}{u} \quad \textit{for every } x \in \mathbb{R}^N.$$

Proof. It is a consequence of the assumption $p, q > 1$. □

In what follows we consider $H_1, H_2 : \mathbb{R}^{N+1} \to \mathbb{R}$ given by

$$H_1(x,v) = \frac{w_\lambda^{p+1} - \big(v_{1,\lambda}+1\big)^{p+1}}{(p+1)} - \big(v_{1,\lambda}+1\big)^p v^+$$

and

$$H_2(x,u) = \frac{z_\lambda^{q+1} - \big(u_{1,\lambda}+1\big)^{q+1}}{(q+1)} - \big(u_{1,\lambda}+1\big)^q u^+,$$

where $z_\lambda = u_{1,\lambda} + u^+ + 1$ and $w_\lambda = v_{1,\lambda} + v^+ + 1$.

Remark 3.3.1. From definition of H_1 and H_2 we have

$$\begin{aligned}H(x,u,v) &= \mathcal{H}(u_{1,\lambda}+u^+, v_{1,\lambda}+v^+) - \mathcal{H}(u_{1,\lambda}, v_{1,\lambda}) - \big(h_1(v_{1,\lambda})v^+ + h_2(u_{1,\lambda})u^+\big)\\ &= H_1(x,v) + H_2(x,u).\end{aligned}$$

Moreover defining $l : [0,\infty) \to \mathbb{R}$ by

$$l(t) = (v_{1,\lambda} + tv^+ + 1)^{p+1},$$

there exist $\xi \in (0,1)$ such that

$$\begin{aligned}(v_{1,\lambda}+v^+ + 1)^{p+1} - (v_{1,\lambda}+1)^{p+1} &= l(1) - l(0)\\ &= l'(\xi) = (p+1)(v_{1,\lambda} + \xi v^+ + 1)^p v^+\\ &\geq (p+1)(v_{1,\lambda}+1)^p v^+.\end{aligned}$$

This implies that $H_1(x,v) \geq 0$ for every $x \in \mathbb{R}^N$ and $v \in \mathbb{R}$. Similarly $H_2(x,u) \geq 0$ for every $x \in \mathbb{R}^N$ and $u \in \mathbb{R}$. Therefore

$$H(x,u,v) \geq 0 \;\text{ for all } x \in \mathbb{R}^N \;\text{ and every }\; u,v \in \mathbb{R}.$$

The following lemma holds.

Lemma 3.3.3. *If $q > 1$, then we have*

$$\lim_{v\to\infty} \frac{H_1(x,v)}{v^2} = \infty = \lim_{u\to\infty} \frac{H_2(x,u)}{u^2} \quad \textit{for every } x \in \mathbb{R}^N.$$

The following result is crucial in our approach.

Lemma 3.3.4. *If $q > 1$, then there exist constants $t_1, t_2 \in \left(\frac{N}{2}, N\right)$, $C_0 > 0$ and $R_0 > 0$ such that*

$$C_0 f(x,v)^{t_1} \leq v^{t_1}\left(\frac{f(x,v)v}{2} - H_1(x,v)\right) \quad \text{and} \quad C_0 g(x,u)^{t_2} \leq u^{t_2}\left(\frac{g(x,u)u}{2} - H_2(x,u)\right),$$

for all $x \in \mathbb{R}^N$ and $u, v \geq R_0$.

Proof. Let $x \in \mathbb{R}^N$ and $u, v \geq 0$. We have

$$\begin{aligned}\frac{f(x,v)v}{2} - H_1(x,v) &= \frac{w_\lambda^p v + (v_{1,\lambda}+1)^p v}{2} - \frac{w_\lambda^{p+1} - (v_{1,\lambda}+1)^{p+1}}{p+1}\\ &\approx \frac{p-1}{2(p+1)} v^{p+1} \;\text{ for }\; v \approx \infty.\end{aligned}$$

On the other hand,

$$\begin{aligned}f(x,v)^{t_1} &= \left(w_\lambda^p - (v_{1,\lambda}+1)^p\right)^{t_1}\\ &\approx v^{pt_1} \;\text{ for }\; v \approx \infty.\end{aligned}$$

Thus, for sufficiently large values of v the following inequality holds

$$\frac{v^{t_1}\left(\frac{f(x,v)v}{2} - H_1(x,v)\right)}{f(x,v)^{t_1}} \geq C_0 > 0$$

if and only if

$$\frac{(p-1)}{2(p+1)}\frac{v^{p+t_1+1}}{v^{pt_1}} \geq C_0 > 0$$

or equivalently

$$t_1 \leq \frac{p+1}{p-1}.$$

Since $1 < p < 2^* - 1$ we see that

$$\frac{N}{2} \leq \frac{p+1}{p-1}.$$

Therefore we can choose $t_1 \in \left(\frac{N}{2}, N\right)$. The existence of $t_2 \in \left(\frac{N}{2}, N\right)$ follows analogously. This concludes the proof of the lemma. □

As a consequence of **Lemma 3.3.2**, and **Lemma 3.3.4** we have

Lemma 3.3.5. *If $q > 1$, then we have*

$$\lim_{v\to\infty}\left(\frac{f(x,v)v}{2} - H_1(x,v)\right) = \infty = \lim_{u\to\infty}\left(\frac{f(x,u)u}{2} - H_2(x,u)\right) \quad \textit{for every } x \in \mathbb{R}^N.$$

Remark 3.3.2. For the purpose of applying **Theorem 1.6.7** we note that we can decompose $E = E^+ \oplus E^-$, where

$$E^+ := \{(u,u)\ ;\ u \in H^1_V(\mathbb{R}^N)\},\quad E^- := \{(u,-u)\ ;\ u \in H^1_V(\mathbb{R}^N)\}$$

and both spaces are infinite dimensional.
Also, one can easily check that for any $z = (u,v) \in E,\ z = z^+ + z^-$ with

$$z^+ = \left(\frac{u+v}{2}, \frac{u+v}{2}\right),\quad z^- = \left(\frac{u-v}{2}, -\frac{u-v}{2}\right)$$

and

$$\int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)dx = \frac{1}{2}\big(\|z^+\|^2 - \|z^-\|^2\big).$$

Then we can write

$$I_\lambda(u,v) = \frac{1}{2}\big(\|z^+\|^2 - \|z^-\|^2\big) - \Phi(u,v)$$

with $\Phi(u,v) = \lambda \int_{\mathbb{R}^N} \rho(x)H(x,u,v)dx$.

Lemma 3.3.6.

i) There exists $\lambda_1^* > 0$ small enough and $r_0,\ a > 0$, such that $I_\lambda|_{N_r} \geq a$ for every $\lambda \in (0, \lambda_1^*)$.

ii) For r_0 given in i) and any $z_0 = (u_0,u_0) \in E^+ \setminus \{0\}$ with $\|z_0\| = 1$, there is $R > r_0$ such that $I_\lambda|_{\partial M_{R,z_0}} \leq 0$.

Proof.

i) For any $z \in N_{r_0}$, there exists $u \in H^1_V(\mathbb{R}^N)$ with $z = (u, u)$ and $\|z\| = r_0$. Then

$$I_\lambda(z) = \|z\|^2 - \lambda \int_{\mathbb{R}^N} \rho(x) H(x, u, u) dx$$

Therefore, from **(3.3.2)** and **Proposition 1.2.3**, we see that there are $0 < \lambda_1^* < \lambda^*$ and $C_1 > 0$ such that for every $0 < \lambda < \lambda_1^*$, if $\|z\| = r_0 = \lambda$, then

$$I_\lambda(z) \geq \lambda^2 (1 - 2\lambda C_1) := a_\lambda > 0.$$

ii) Let $z \in \partial M_R$, then $z = z^- + tz_0$ with $\|z\| = R$, $t > 0$ or $\|z\| < R$, $t = 0$.

a) First suppose $t = 0$. Then, we have $z \in E^-$, that is, $z = (u, -u)$ and

$$I_\lambda(z) = I_\lambda(u, -u) = -\frac{1}{2}\|z^-\|^2 - \lambda \int_{\mathbb{R}^N} \rho(x) H(x, u, -u) dx \leq 0$$

since $\rho(x) > 0$ and $H(x, u, -u) \geq 0$, for any $x \in \mathbb{R}^N$, $u \in \mathbb{R}$.

b) Now, assume that $t > 0$. Let us suppose by contradiction, that there is a sequence $(z_n) \subset \partial M_{R_n}$, with

$$z_n = t_n z_0 + z_n^-, \quad t_n > 0, \quad \|z_0\| = 1 \quad \text{and} \quad \|z_n\| = R_n \to \infty$$

such that $I_\lambda(z_n) > 0$, i.e., if $z_n = (u_n, v_n) = (t_n u_0 + \phi_n, t_n u_0 - \phi_n)$, then

$$I_\lambda(z_n) = I_\lambda(u_n, v_n) = \frac{1}{2}\Big(t_n^2\|z_0\|^2 - \|z_n^-\|^2\Big) - \lambda \int_{\mathbb{R}^N} \rho(x) H(x, u_n, v_n) dx > 0.$$

Denote $\delta_n = \dfrac{t_n}{\|z_n\|}$ and $w_n^- = \dfrac{z_n^-}{\|z_n\|}$. Then

$$\frac{I_\lambda(z_n)}{\|z_n\|^2} = \frac{1}{2}\Big(\delta_n^2 - \|w_n^-\|^2\Big) - \lambda \int_{\mathbb{R}^N} \rho(x) \frac{H(x, u_n, v_n)}{\|z_n\|^2} dx > 0. \tag{3.3.3}$$

Since $\rho(x) > 0$ and $G(x, u, v) \geq 0$ for any $x \in \mathbb{R}^N$ and $u, v \in \mathbb{R}$, we have

$$\delta_n \geq \|w_n^-\|. \tag{3.3.4}$$

Moreover, notice that

$$\delta_n^2 + \|w_n^-\|^2 = \frac{t_n^2\|z_0\|^2}{\|z_n\|^2} + \frac{\|z_n^-\|^2}{\|z_n\|^2} = 1. \tag{3.3.5}$$

And then it follows from **(3.3.4)** and **(3.3.5)** that

$$\frac{1}{\sqrt{2}} \leq \delta_n \leq 1 \tag{3.3.6}$$

and w_n^- is bounded. Going to a subsequence, we may assume that $\delta_n \to \delta$ for some $\delta > 0$ and $w_n^- \rightharpoonup w^- = (\phi, -\phi)$ in E as $n \to \infty$.

Thus $\dfrac{t_n^2}{\|z_n\|^2} \to \delta^2 > 0$ and since $\|z_n\| \to \infty$ it follows that $t_n \to \infty$.
Moreover

$$\frac{u_n}{\|z_n\|} = \frac{t_n u_0 + \phi_n}{\|z_n\|} \rightharpoonup \delta u_0 + \phi \quad \text{and} \quad \frac{v_n}{\|z_n\|} = \frac{t_n u_0 - \phi_n}{\|z_n\|} \rightharpoonup \delta u_0 - \phi$$

in E. Hence, by **Proposition 1.2.3** we may assume, up to a subsequence that

$$\frac{u_n(x)}{\|z_n\|} = \frac{t_n u_0(x) + \phi_n(x)}{\|z_n\|} \to \delta u_0(x) + \phi(x) \ \text{ a.e. in } \ \mathbb{R}^N$$

and

$$\frac{v_n(x)}{\|z_n\|} = \frac{t_n u_0(x) - \phi_n(x)}{\|z_n\|} \to \delta u_0(x) - \phi(x) \ \text{ a.e. in } \ \mathbb{R}^N$$

Let us denote $A_1 = \{x \in \mathbb{R}^N \ ; \ \delta u_0(x) + \phi(x) \neq 0\}$. We have

$$\lim_{n\to\infty} \frac{t_n u_0(x) + \phi_n(x)}{\|z_n\|} = \delta u_0(x) + \phi(x) \neq 0, \quad \text{a.e. in } \ A_1$$

which means that

$$u_n(x) = t_n u_0(x) + \phi_n(x) \to \infty \text{ a.e. in } A_1.$$

Analogously, if we denote $A_2 = \{x \in \mathbb{R}^N \ ; \ \delta u_0(x) - \phi(x) \neq 0\}$, then we have

$$v_n(x) = t_n u_0(x) - \phi_n(x) \to \infty \text{ a.e. in } A_2.$$

On the other hand, we notice that

$$\lim_{u\to\infty} \frac{H_2(x,u)}{u^2} = \begin{cases} 0 & \text{if} \quad 0 < q < 1 \\ \frac{1}{2} & \text{if} \quad q = 1. \end{cases} \tag{3.3.7}$$

From **Lemma 3.3.3**, (3.3.3), (3.3.5), (3.3.7), using Fatou's lemma and the fact that H_1, H_2 are positive functions, we obtain

$$\begin{aligned}
0 \ &\leq \ \frac{1}{2}\Big(\delta^2 - \|w^-\|^2\Big) - \lambda \liminf_{n\to\infty} \int_{\mathbb{R}^N} \rho(x) \frac{H(x,u_n,v_n)}{\|z_n\|^2} dx \\
&= \ \frac{1}{2}\Big(\delta^2 - \|w^-\|^2\Big) - \lambda \liminf_{n\to\infty} \int_{\mathbb{R}^N} \rho(x) \frac{H_1(x,v_n) + H_2(x,u_n)}{\|z_n\|^2} dx \\
&= \ \frac{1}{2}\Big(\delta^2 - \|w^-\|^2\Big) - \lambda \liminf_{n\to\infty} \int_{\mathbb{R}^N} \rho(x) \left(\frac{H_1(x,v_n)}{v_n^2} \frac{v_n^2}{\|z_n\|^2} + \frac{H_2(x,u_n)}{u_n^2} \frac{u_n^2}{\|z_n\|^2} \right) dx \\
&\leq \ \frac{1}{2}\Big(\delta^2 - \|w^-\|^2\Big) - \lambda \int_{\mathbb{R}^N} \rho(x) \liminf_{n\to\infty} \left(\frac{H_1(x,v_n)}{v_n^2} \frac{v_n^2}{\|z_n\|^2} + \frac{H_2(x,u_n)}{u_n^2} \frac{u_n^2}{\|z_n\|^2} \right) dx \\
&= \ \frac{1}{2}\Big(\delta^2 - \|w^-\|^2\Big) - \lambda \int_{A_1} \rho(x) \liminf_{n\to\infty} \left(\frac{H_1(x,v_n)}{v_n^2} \frac{v_n^2}{\|z_n\|^2} \right) dx \\
&- \ \lambda \int_{A_2} \rho(x) \liminf_{n\to\infty} \left(\frac{H_2(x,u_n)}{u_n^2} \frac{u_n^2}{\|z_n\|^2} \right) dx = -\infty,
\end{aligned}$$

which is a contradiction and the lemma is proved.

□

The following result is a key point in our argument to obtain a second solution to the Hamiltonian system.

Lemma 3.3.7. *Let $(z_n) \subset E$ is a $(C)_c$-sequence of I_λ. Then (z_n) is bounded in E, for sufficiently small values of λ.*

Proof. Here taking sufficiently small λ we managed to adapt the proof of [42, Lemma 3.2]. In fact, let $(z_n) \subset E$ be a $(C)_c$-sequence of I_λ. Then

$$I_\lambda(z_n) \to c \ \text{ and } \ I'_\lambda(z_n)z_n \to 0. \tag{3.3.8}$$

We denote $z_n = (u_n, v_n)$. We may assume, by contradiction, that $\|z_n\| \to \infty$. We set

$$w_n = \frac{z_n}{\|z_n\|} = \left(\frac{u_n}{\|z_n\|}, \frac{v_n}{\|z_n\|}\right) := (w_n^1, w_n^2).$$

Then (w_n) is bounded in E with $\|w_n\| = 1$. Notice that

$$I'_\lambda(z_n)(z_n^+ - z_n^-) = I'_\lambda(u_n, v_n)(v_n, u_n) = \|z_n\|^2 - \lambda \int_{\mathbb{R}^N} \rho(x)\Big(f(x, v_n)u_n + g(x, u_n)v_n\Big)dx.$$

Thus

$$\frac{I'_\lambda(z_n)(z_n^+ - z_n^-)}{\|z_n\|^2} = 1 - \lambda \int_{\mathbb{R}^N} \rho(x)\left(\frac{f(x, v_n)u_n}{\|z_n\|^2} + \frac{g(x, u_n)v_n}{\|z_n\|^2}\right) dx. \tag{3.3.9}$$

Since $\|z_n\| = \|z_n^+ - z_n^-\|$, it follows by Cerami condition and (3.3.9) that

$$\lim_{n\to\infty} \lambda \int_{\mathbb{R}^N} \rho(x)\left(\frac{f(x, v_n)u_n}{\|z_n\|^2} + \frac{g(x, u_n)v_n}{\|z_n\|^2}\right) dx = 1. \tag{3.3.10}$$

Let $0 \leq a < b \leq +\infty$ and define

$$A_n(a, b) = \{x \in \mathbb{R}^N \ ; \ a \leq v_n(x) < b\}.$$

Now we will work with this set.

i) Using the definition of f, there is $a > 0$ small enough such that $f(x, v) \leq Cv$ for each $0 \leq v \leq a$, uniformly in $x \in \mathbb{R}^N$, then, for any $n \in \mathbb{N}$, we have

$$\begin{aligned}
\int_{A_n(0,a)} \rho(x)\frac{f(x, v_n)u_n}{\|z_n\|^2}dx &\leq C \int_{A_n(0,a)} \rho(x)\frac{v_n u_n}{\|z_n\|^2}dx \\
&= C \int_{A_n(0,a)} \rho(x)w_n^1 w_n^2 dx \\
&\leq C \left(\int_{A_n(0,a)} \rho(x)|w_n^1|^2 dx\right)^{\frac{1}{2}} \left(\int_{A_n(0,a)} \rho(x)|w_n^2|^2 dx\right)^{\frac{1}{2}} \\
&\leq C\|w_n^1\|_{H_V^1(\mathbb{R}^N)}\|w_n^2\|_{H_V^1(\mathbb{R}^N)} \\
&\leq C.
\end{aligned}$$

ii) By **Lemma 3.3.4**, we have

$$\frac{f(x,v)v}{2} - H_1(x,v) \approx \frac{p-1}{2(p+1)} v^{p+1} \quad \text{for} \quad v \approx \infty \quad \text{and every } x \in \mathbb{R}^N. \tag{3.3.11}$$

Thus, and **(3.3.8)** implies that for n and b sufficiently large

$$\begin{aligned} c + o(1) = I_\lambda(z_n) - \frac{1}{2} I'_\lambda(z_n) z_n &\geq \lambda \int_{A_n(b,+\infty)} \rho(x) \left(\frac{f(x,v_n)v_n}{2} - H_1(x,v_n) \right) dx \\ &\geq \lambda \inf_{|v| \geq b} \left(\frac{f(x,v)v}{2} - H_1(x,v) \right) \int_{A_n(b,+\infty)} \rho(x) dx. \end{aligned}$$

Again using **(3.3.11)** it follows that, for n sufficiently large,

$$\int_{A_n(b,+\infty)} \rho(x) dx \to 0, \text{ as } b \to +\infty.$$

Let $t_1 \in \left(\frac{N}{2}, N\right)$ given by **Lemma 3.3.4**. Using the fact that $\|w_n^1\|_{H^1_V(\mathbb{R}^N)} \leq 1$, for $s_1 = \dfrac{1}{\frac{1}{2} + \frac{1}{N} - \frac{1}{t_1}}$ and n sufficiently large, we obtain

$$\begin{aligned} \int_{A_n(b,+\infty)} \rho(x) |w_n^1|^{s_1} dx &= \int_{A_n(b,+\infty)} \rho^{\frac{2^*-s_1}{2^*}}(x) \rho^{\frac{s_1}{2^*}}(x) |w_n^1|^{s_1} dx \\ &\leq \left(\int_{A_n(b,+\infty)} \rho(x) dx \right)^{\frac{2^*-s_1}{2^*}} \left(\int_{A_n(b,+\infty)} \rho(x) |w_n^1|^{2^*} dx \right)^{\frac{s_1}{2^*}} \\ &\leq C \left(\int_{A_n(b,+\infty)} \rho(x) dx \right)^{\frac{2^*-s_1}{2^*}} \to 0, \text{ as } b \to +\infty. \end{aligned}$$

Thus, for n sufficiently large, using generalized Hölder's inequality we have

$$\begin{aligned} \int_{A_n(b,+\infty)} \rho(x) \frac{f(x,v_n)u_n}{\|z_n\|^2} dx &= \int_{A_n(b,+\infty)} \rho^{\frac{1}{\tau_1}}(x) \rho^{\frac{1}{s_1}}(x) \rho^{\frac{1}{2^*}}(x) \frac{f(x,v_n)}{v_n} \frac{v_n}{\|z_n\|} \frac{u_n}{\|z_n\|} dx \\ &= \int_{A_n(b,+\infty)} \rho^{\frac{1}{\tau_1}}(x) \rho^{\frac{1}{s_1}}(x) \rho^{\frac{1}{2^*}}(x) \frac{f(x,v_n)}{v_n} w_n^1 w_n^2 dx \\ &\leq \left(\int_{A_n(b,+\infty)} \rho(x) \left(\frac{|f(x,v_n)|}{|v_n|} \right)^{\tau_1} dx \right)^{\frac{1}{\tau_1}} \\ &\quad \cdot \left(\int_{A_n(b,+\infty)} \rho(x) |w_n^1|^{s_1} dx \right)^{\frac{1}{s_1}} \left(\int_{A_n(b,+\infty)} \rho(x) |w_n^2|^{2^*} dx \right)^{\frac{1}{2^*}} \\ &\leq C \left(\int_{A_n(b,+\infty)} \rho(x) \left(\frac{f(x,v_n)v_n}{2} - H_1(x,v_n) \right) dx \right)^{\frac{1}{\tau_1}} \\ &\quad \cdot \left(\int_{A_n(b,+\infty)} \rho(x) |w_n^1|^{s_1} dx \right)^{\frac{1}{s_1}} \\ &\leq C \left(\int_{A_n(b,+\infty)} \rho(x) |w_n^1|^{s_1} dx \right)^{\frac{1}{s_1}} \to 0, \text{ as } b \to +\infty. \end{aligned}$$

iii) Note that there is a constant $C > 0$ independent of n, but depending on a and b, such that

$$|f(x, v_n)| \leq C|v_n|, \quad \text{for all } x \in A_n(a, b). \tag{3.3.13}$$

On the other hand, from the hypotheses on V and ρ, we have

$$\frac{V(x)}{\rho(x)} \geq \frac{a\big(1 + |x|^{\beta}\big)}{k\big(1 + |x|^{\alpha}\big)}, \quad \alpha \in (0, 2], \quad \alpha + \beta > 4.$$

This implies that there is a $R_0 > 0$ sufficiently large such

$$\frac{f(x, v)}{v} \leq \frac{V(x)}{\rho(x)}, \quad \text{for } v \in (a, b) \text{ and } |x| > R_0.$$

Thus, by **(3.3.13)** we have

$$\begin{aligned}
\int_{\substack{A_n(a,b)\\ |x|>R_0}} \rho(x)\frac{f(x, v_n)u_n}{\|z_n\|^2}dx &= \int_{\substack{A_n(a,b)\\ |x|>R_0}} \rho(x)\frac{f(x, v_n)}{v_n}w_n^2 w_n^1 dx \\
&\leq \int_{\substack{A_n(a,b)\\ |x|>R_0}} \eta V(x) w_n^2 w_n^1 dx \leq \lambda \int_{\mathbb{R}^N} \eta V(x) w_n^2 w_n^1 dx \\
&\leq \left(\int_{\mathbb{R}^N} V(x)|w_n^1|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} V(x)|w_n^2|^2 dx\right)^{\frac{1}{2}} \\
&\leq \|w_n^1\|_{H_V^1(\mathbb{R}^N)} \|w_n^2\|_{H_V^1(\mathbb{R}^N)} \leq 1 \quad \forall n \in \mathbb{N}.
\end{aligned}$$

Furthermore, we see that

$$\int_{\substack{A_n(a,b)\\ |x|\leq R_0}} \rho(x)|w_n^2|^2 dx = \frac{1}{\|z_n\|^2}\int_{\substack{A_n(a,b)\\ |x|\leq R_0}} \rho(x) v_n^2 dx \leq \frac{b^2}{\|z_n\|^2}\int_{\substack{A_n(a,b)\\ |x|\leq R_0}} \rho(x) dx \to 0$$

as $n \to \infty$. Consequently, we have

$$\begin{aligned}
\int_{\substack{A_n(a,b)\\ |x|\leq R_0}} \rho(x)\frac{f(x, v_n)u_n}{\|z_n\|^2}dx &= \int_{\substack{A_n(a,b)\\ |x|\leq R_0}} \rho(x)\frac{f(x, v_n)w_n^1}{\|z_n\|}dx \leq C\int_{\substack{A_n(a,b)\\ |x|\leq R_0}} \rho(x)\frac{v_n w_n^1}{\|z_n\|}dx \\
&= C\int_{\substack{A_n(a,b)\\ |x|\leq R_0}} \rho(x) w_n^1 w_n^2 dx \\
&\leq C\|w_n^1\|_{H_V^1(\mathbb{R}^N)} \left(\int_{\substack{A_n(a,b)\\ |x|\leq R_0}} \rho(x)|w_n^2|^2 dx\right)^{1/2} \to 0.
\end{aligned}$$

Therefore we obtain

$$\int_{A_n(a,b)} \rho(x)\frac{f(x, v_n)u_n}{\|z_n\|^2}dx \leq 1 \quad \text{for } n \text{ sufficiently large.}$$

Finally by i) ii) and iii) we conclude that

$$\int_{\mathbb{R}^N} \rho(x)\frac{f(x,v_n)u_n}{\|z_n\|^2}dx \leq 1+C \quad \text{for } n \text{ sufficiently large.}$$

If $q>1$, we can use a similar argument to prove

$$\int_{\mathbb{R}^N} \rho(x)\frac{g(x,u_n)v_n}{\|z_n\|^2}dx \leq 1+C \text{ for } n \text{ sufficiently large.}$$

Now, suppose $0<q\leq 1$. Since $g(x,u)\leq u$ for each $u\geq 0$, uniformly in $x\in\mathbb{R}^N$, proceeding as in the case i) we have that

$$\int_{\mathbb{R}^N} \rho(x)\frac{g(x,u_n)v_n}{\|z_n\|^2}dx \leq \int_{\mathbb{R}^N} \rho(x)\frac{v_n u_n}{\|z_n\|^2}dx \leq C \quad \text{for all } n\in\mathbb{N}.$$

Therefore, we get

$$\int_{\mathbb{R}^N} \rho(x)\left(\frac{f(x,v_n)u_n}{\|z_n\|^2}+\frac{g(x,u_n)v_n}{\|z_n\|^2}\right)dx \leq 2(1+C) \quad \text{for } n \text{ sufficiently large.}$$

If we consider $2\big(1+C\big)\lambda<1$, this fact contradicts **(3.3.10)**. Therefore, (z_n) is bounded in E and the lemma is proved. □

Lemma 3.3.8. *Let $(u_n,v_n)\subset E$ be a bounded sequence such that $(u_n,v_n)\rightharpoonup(u,v)$ in E. Then*

$$\int_{\mathbb{R}^N} \rho(x)f(x,v_n)u_n dx \to \int_{\mathbb{R}^N} \rho(x)f(x,v)u dx \tag{3.3.14}$$

and

$$\int_{\mathbb{R}^N} \rho(x)g(x,u_n)v_n dx \to \int_{\mathbb{R}^N} \rho(x)g(x,u)v dx. \tag{3.3.15}$$

Proof. As in the **Lemma 3.3.1**, since $p>1$ and $p,q<2^*-1$, there exists $C>0$ such that

$$f(x,v)\leq C(v+v^p) \quad \text{and} \quad g(x,u)\leq \begin{cases} u & \text{if} \quad 0<q\leq 1 \\ C(u+u^q) & \text{if} \quad q>1, \end{cases}$$

for all $x\in\mathbb{R}^N$ and every $u,v\geq 0$. So, using **Proposition 1.2.3** and standard arguments we get **(3.3.14)** and **(3.3.15)**.

□

To finish this section, we show the existence of a solution of System **(HS_A^λ)**. As we mentioned earlier, for this, we will use **Theorem 1.6.7**. Indeed, by **Remark 3.3.2** for $z=(u,v)\in E$ we have

$$I_\lambda(z)=\frac{1}{2}\big(\|z^+\|^2-\|z^-\|^2\big)-\Phi(z)$$

with

$$\Phi(z)=\Phi(u,v)=\lambda\int_{\mathbb{R}^N}\rho(x)H(x,u,v)dx.$$

We notice that $\Phi\in C^1\big(E,\mathbb{R}\big)$ and $\Phi(z)\geq 0$. By Fatou's lemma Φ is weakly lower semicontinuous and Φ' is weakly sequentially continuous. Moreover, **Lemma 3.3.6**, gives us the existence

of $0 < \lambda^{**} \leq \lambda^*$ small enough, so that for every $\lambda \in (0, \lambda^{**})$, there are $r_0 > 0$ and $a > 0$, such that $I_\lambda|_{N_{r_0}} \geq a$. Also, for such r_0, there exist $R > r_0$ and $z_0 \in E^+ \setminus \{0\}$ with $\|z_0\| = 1$ such that $I_\lambda|_{\partial M_{R,z_0}} \leq 0$.
Thus, by **Theorem 1.6.7** there exists a $(C)_c$-sequence $(z_n) \subset E$ for I_λ which is bounded in E by **Lemma 3.3.7**. Then, up to a subsequence, we may assume that $z_n \rightharpoonup z$ in E.

We denote $z_n = (u_n, v_n)$ and $z = (u, v)$. By **Lemma 3.3.8** and since $I'_\lambda(z_n)(z_n) \to 0$, as $n \to \infty$, we have

$$\lim_{n\to\infty} \|z_n\|^2 = \lambda \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} \rho(x) f(x, v_n) u_n dx + \int_{\mathbb{R}^N} \rho(x) g(x, u_n) v_n dx \right)$$
$$= \lambda \left(\int_{\mathbb{R}^N} \rho(x) f(x, v) u dx + \int_{\mathbb{R}^N} \rho(x) g(x, u) v dx \right).$$

Also, using that $I'_\lambda(u_n, v_n)(v, u) \to 0$, as $n \to \infty$, we have

$$\begin{aligned}
\|z\|^2 &= \|u\|^2_{H^1_V(\mathbb{R}^N)} + \|v\|^2_{H^1_V(\mathbb{R}^N)} \\
&= \lim_{n\to\infty} \Big(\langle u_n, u\rangle_{H^1_V(\mathbb{R}^N)} + \langle v_n, v\rangle_{H^1_V(\mathbb{R}^N)} \Big) \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} \Big(\nabla u_n \nabla u + V(x) u_n u + \nabla v_n \nabla v + V(x) v_n v \Big) dx \\
&= \lambda \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} \rho(x) f(x, v_n) u dx + \int_{\mathbb{R}^N} \rho(x) g(x, u_n) v dx \right) \\
&= \lambda \left(\int_{\mathbb{R}^N} \rho(x) f(x, v) u dx + \int_{\mathbb{R}^N} \rho(x) g(x, u) v dx \right).
\end{aligned}$$

Hence

$$\lim_{n\to\infty} \|z_n\|^2 = \|z\|^2.$$

Which shows that

$$z_n \to z \quad \text{in } E.$$

Therefore, $z = (u, v)$ is a nontrivial solution of problem **(HS^λ_A)** with $I_\lambda(u, v) = c \geq a > 0$. Moreover, by maximum principle $u > 0$ and $v > 0$.
Therefore $(u_{1,\lambda} + u, v_{1,\lambda} + v)$ is a positive solution of System **(HS_λ)**. This concludes the proof of **Theorem 5** part iii).

3.4 Some nonhomogeneous elliptic system

To conclude this chapter, we give an application of **Theorem 1**. For this purpose, let us introduce the following System, which, in part, has motivated our study :

$$\begin{cases} -\Delta z = \rho_1(x) z^r w^p & \text{in} \quad \mathbb{R}^N \\ -\Delta w = \rho_2(x) z^q w^s & \text{in} \quad \mathbb{R}^N, \\ z(x) \to c_1, w(x) \to c_2 & \text{as} \quad |x| \to \infty \end{cases} \tag{3.4.1}$$

where ρ_i satisfies (H_ρ) with $\beta > 2$ with $c_1, c_2 \geq 0$ to be fixed later. Note that the solutions of this System do not belong to any Sobolev space, so it is difficult to solve directly. However, as we will see in the last section, a strategy involving **Theorem 1** allows us to find a solution of System **(3.4.1)**, which apparently is the only way to solve it.

We claim that system **(3.4.1)** has a bounded positive solution $(\bar{z}, \bar{w})$ in the following two cases:

i) $p, q > 0$ and $r, s > 1$.

ii) $r = s = 0$ and $0 < pq < 1$.

In fact, for $\lambda,\ \mu > 0$ sufficiently small, **Theorem 1** guarantees the existence of a positive bounded solution $(\tilde{u}, \tilde{v})$ of the system

$$\begin{cases} -\Delta u = \lambda \rho_1(x)(u+1)^r (v+1)^p & \text{in} \quad \mathbb{R}^N \\ -\Delta v = \mu \rho_2(x)(u+1)^q (v+1)^s & \text{in} \quad \mathbb{R}^N, \\ u(x), v(x) \to 0 & \text{as} \quad |x| \to \infty. \end{cases} \tag{3.4.2}$$

Now, we will find a condition for $c_1,\ c_2 \geq 0$ such that the pair $(\bar{z}, \bar{w}) := (c_1(\tilde{u}+1), c_2(\tilde{v}+1))$ be a bounded solution of System **(3.4.2)**. Indeed, with respect to case $i)$ we have

$$-\Delta \bar{z} = -c_1 \Delta \tilde{u} = c_1 \lambda \rho_1(x)(\tilde{u}+1)^r (\tilde{v}+1)^p = c_1^{1-r} c_2^{-p} \lambda \rho_1(x) \bar{z}^r \bar{w}^p$$

and

$$-\Delta \bar{w} = -c_2 \Delta \tilde{v} = c_2 \mu \rho_2(x)(\tilde{u}+1)^q (\tilde{v}+1)^s = c_1^{-q} c_2^{1-s} \mu \rho_2(x) \bar{z}^r \bar{w}^p.$$

Thus for $c_1,\ c_2$ small enough we have $\lambda = c_1^{r-1} c_2^{p} < \Lambda$ and $\mu = c_1^{q} c_2^{s-1} < \Lambda$, and so System **(3.4.2)** has a positive bounded solution.

Related to the $ii)$, we choose $\gamma > q$ such that $p\gamma < 1$, $c_1 = c$ and $c_2 = c^\gamma$. Then, similar to case $i)$ we see that

$$-\Delta \bar{z} = c^{1-p\gamma} \lambda \rho_1(x) \bar{w}^p \quad \text{and} \quad -\Delta \bar{w} = c^{\gamma - q} \mu \rho_2(x) \bar{z}^q.$$

Thus, $\lambda = c^{p\gamma-1}$ and $\mu = c^{q-\gamma}$ verify $0 < \lambda,\ \mu < \Lambda$ for a sufficiently large c. Hence the pair $(\bar{z}, \bar{w})$ is a bounded positive solution of System

$$\begin{cases} -\Delta z = \rho_1(x) w^p & \text{in} \quad \mathbb{R}^N \\ -\Delta w = \rho_2(x) z^q & \text{in} \quad \mathbb{R}^N, \\ z(x) \to c, w(x) \to c^\gamma & \text{as} \quad |x| \to \infty. \end{cases} \tag{3.4.3}$$

Finally, we would like to mention the paper [35], where this type of problems was studied with $c = 0$ (see [35, Theorem 5.1]).

Chapter 4

The linear equation in the half space

In this Chapter, we will develop some results obtained, which are still under development, in which we will give sufficient and necessary conditions to obtain the existence of bounded solutions of Poisson's equation in the half space:

$$-\Delta u = \rho(x) \text{ in } \mathbb{R}^N_+, \tag{4.1}$$

where $\rho \in L^\infty_{loc}\left(\mathbb{R}^N_+\right)$, $\rho(x) \geq 0$ and ρ not identically zero.
Our focus is on obtaining solutions of **(4.1)** that vanishing at infinity as follows

$$\liminf_{|x|\to\infty} u(x) = 0 \quad \text{and} \quad \liminf_{x_N\to 0} u(x) = 0. \tag{4.2}$$

For this purpose, we notice that if $u \in C^2(\overline{\mathbb{R}^N_+})$ solves the problem

$$\begin{cases} -\Delta u = \rho(x) & \text{in} \quad \mathbb{R}^N_+ \\ u = g & \text{on} \quad \partial\mathbb{R}^N_+, \end{cases}$$

for any continuous boundary values g, then u is given by

$$u(x) = \int_{\mathbb{R}^N_+} G_{\mathbb{R}^N_+}(x,y)\rho(y)dy - \frac{2x_N}{nw_N}\int_{\partial\mathbb{R}^N_+} \frac{g(y)}{|x-y|^N} dS(y) \qquad \forall x \in \mathbb{R}^N_+,$$

where

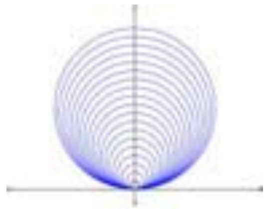
$$G_{\mathbb{R}^N_+}(x,y) = \Gamma(x-y) - \Gamma(x-\tilde{y}) \text{ for all } x \neq y \text{ in } \mathbb{R}^N_+,$$

is the Green's function in $\mathbb{R}^N_+$. Therefore, we expect that

$$\int_{\mathbb{R}^N_+} G_{\mathbb{R}^N_+}(x,y)\rho(y)dy \qquad \forall x \in \mathbb{R}^N_+,$$

will be a bounded solution of Problem **(4.1)**.

As we saw in the introduction, to obtain sufficient and necessary conditions to obtain the existence of solutions of Problem (4.1), we will use a monotonicity argument involving Green's function in the half space and the Green's function in the balls $B_n(a_n) = \{x \in \mathbb{R}^N : |x - a_n| < n\}$, where $a_n = (0, \ldots, 0, n) \in \mathbb{R}^N_+$, since the half space is the infinite union of these balls:



$$\mathbb{R}^N_+ = \bigcup_{n=1}^{\infty} B_n(a_n).$$

For this, first we note that, for $y \in B(a, R)$, if we define

$$\phi^y(x) = \begin{cases} \Gamma\left(\dfrac{|y-a|}{R}\left|x - a - \dfrac{R^2}{|y-a|^2}(y-a)\right|\right) & \text{if} \quad y \neq a \\ \Gamma(R) & \text{if} \quad y = a, \end{cases}$$

where $R > 0$ and $a \in \mathbb{R}^N$. Then $\phi^y \in C^2(B(a, R))$ satisfies

$$\begin{cases} \Delta\phi^y(x) = 0 & \text{in} \quad B(a, R) \\ \phi^y(x) = \Gamma(x - y) & \text{on} \quad \partial B(a, R). \end{cases}$$

Therefore Green's function in $B(a, R)$ is given by

$$G_R(a)(x, y) = G_R(x - a, y - a) \;\text{ for all } x \neq y \;\text{ in }\; B(a, R),$$

where G_R is the Green's function in $B_R = B_R(0)$ given by

$$G_R(x, y) = \begin{cases} \Gamma(x-y) - \Gamma\left(\dfrac{|y|}{R}\left|x - \dfrac{R^2}{|y|^2}y\right|\right) & \text{if} \quad y \neq 0 \\ \Gamma(x) - \Gamma(R) & \text{if} \quad y = 0. \end{cases}$$

for all $x \neq y$ in B_R.

Next, we begin by providing a relationship that exists between the Green's function in $B_n(a_n)$ and the Green's function in $\mathbb{R}^N_+$.

Lemma 4.0.1. *For each $x \in \mathbb{R}^N_+$, the next limit hold:*

$$\lim_{n\to\infty} G_{B_n(a_n)}(x, y) = \Gamma(x - y) - \Gamma(x - \tilde{y}) \;\; \textit{for all} \;\; y \neq x \; \textit{in} \; \mathbb{R}^N_+. \tag{4.3}$$

Proof. Fix $x \in \mathbb{R}^N_+$ and let $y \neq x$ in $\mathbb{R}^N_+$. To prove (4.3) it is enough to show that

$$\lim_{n\to\infty} \frac{|y - a_n|}{n}\left|x - a_n - \frac{n^2}{|y - a_n|^2}(y - a_n)\right| = |x - \tilde{y}|.$$

In fact:

$$\begin{aligned}\frac{|y-a_n|}{n}\left|x-a_n-\frac{n^2}{|y-a_n|^2}(y-a_n)\right| &= \left|\frac{|y-a_n|}{n}(x-a_n)-\frac{n}{|y-a_n|}(y-a_n)\right| \\ &= \left|\frac{|y-a_n|^2(x-a_n)-n^2(y-a_n)}{n|y-a_n|}\right| \\ &= \left|\frac{|y-a_n|^2x-n^2y+a_n\left(n^2-|y-a_n|^2\right)}{n|y-a_n|}\right| \\ &= \left|\frac{|y-a_n|^2x-n^2y+a_n\left(-|y|^2+2ny_N\right)}{n|y-a_n|}\right| \\ &\to |x-y+2y_Ne_N|, \text{ as } n\to\infty \\ &= |x-\tilde{y}|.\end{aligned}$$

This conclude the proof.

□

The following result is a consequence of ρ belongs to $L^\infty_{loc}\left(\mathbb{R}^N_+\right)$.

Lemma 4.0.2. *Let $\rho \in L^\infty_{loc}\left(\mathbb{R}^N_+\right)$, $\rho(x) \geq 0$ and ρ not identically zero. Then, for each $n \in \mathbb{N}$ the linear equation*

$$\begin{cases} -\Delta u = \rho(x) & in \quad B_n(a_n) \\ u = 0 & on \quad \partial B_n(a_n) \end{cases} \tag{$\mathbf{P}_n$}$$

has only one weak positive solution $u_n \in H^1_0(B_n(a_n))$, which increases with n . In addition

$$u_n(x) = \int_{B_n(a_n)} G_{B_n(a_n)}(x,y)\rho(y)dy.$$

Proof. The proof of this lemma is similar to that of **Lemma 2.1.3**. □

Before continuing, we will give some properties of the Green's function in the half space and a technical inequality that will help us throughout this section.

Lemma 4.0.3.

i) There exist $c_1, c_2 > 0$ such that for each $x \neq y \in \mathbb{R}^N_+$:

$$\frac{c_1 x_N y_N}{|x-y|^{N-2}|x-\tilde{y}|^2} \leq G_{\mathbb{R}^N_+}(x,y) \leq \frac{c_2 x_N y_N}{|x-y|^{N-2}|x-\tilde{y}|^2}. \tag{4.4}$$

ii) There exists $c > 0$ such that for each $x \neq y \in \mathbb{R}^N_+$:

$$\frac{c x_N y_N}{|x-\tilde{y}|^N} \leq G_{\mathbb{R}^N_+}(x,y). \tag{4.5}$$

iii) There exists $c > 0$ such that for each $x \neq y \in \mathbb{R}^N_+$:

$$\frac{c x_N y_N}{(|x|+1)^N(|y|+1)^N} \leq G_{\mathbb{R}^N_+}(x,y). \tag{4.6}$$

iv) Let $s > 0$. Then, there exists $c > 0$ such that for $y \in \mathbb{R}^N_+ \backslash B(x,s)$ and each $x \neq y \in \mathbb{R}^N_+$:

$$c|x_i - y_i| \left(\frac{1}{|x-y|^N} - \frac{1}{|x-\tilde{y}|^N} \right) \leq G_{\mathbb{R}^N_+}(x,y), \quad \textit{for } i = 1, \ldots, N. \tag{4.7}$$

v) If $|x-y|^2 \leq x_N y_N$, then

$$\left(\frac{3-\sqrt{5}}{2} \right) x_N \leq y_N \leq \left(\frac{3+\sqrt{5}}{2} \right) x_N. \tag{4.8}$$

Proof.

i) Using the fact

$$|x-\tilde{y}|^2 - |x-y|^2 = 4x_N y_N \quad \text{and} \quad |x-y| \leq |x-\tilde{y}| \quad \text{for all } x, y \in \mathbb{R}^N_+,$$

we get

$$\begin{aligned} \frac{1}{|x-y|^{N-2}} - \frac{1}{|x-\tilde{y}|^{N-2}} &= \frac{1}{|x-y|^{N-2}} \left(1 - \left(\frac{|x-y|}{|x-\tilde{y}|} \right)^{N-2} \right) \\ &\geq \frac{1}{|x-y|^{N-2}} \left(1 - \left(\frac{|x-y|}{|x-\tilde{y}|} \right)^{2} \right) \\ &= \frac{4x_N y_N}{|x-y|^{N-2}|x-\tilde{y}|^2} \quad \text{for all } x \neq y \in \mathbb{R}^N_+ \text{ and } N \geq 4. \end{aligned}$$

Similarly, if $N = 3$ we have

$$\begin{aligned} \frac{1}{|x-y|} - \frac{1}{|x-\tilde{y}|} &= \frac{|x-\tilde{y}| - |x-y|}{|x-y||x-\tilde{y}|} \\ &= \frac{4x_N y_N}{|x-y||x-\tilde{y}|(x-\tilde{y}| + |x-y|)} \\ &\geq \frac{2x_N y_N}{|x-y||x-\tilde{y}|^2} \quad \text{for all } x \neq y \in \mathbb{R}^N_+. \end{aligned}$$

Therefore, there exists $c_1 > 0$ such that

$$\frac{c_1 x_N y_N}{|x-y|^{N-2}|x-\tilde{y}|^2} \leq G_{\mathbb{R}^N_+}(x,y).$$

In the similar way, using that

$$1 - \left(\frac{|x-y|}{|x-\tilde{y}|} \right)^{N-2} \leq \frac{N-2}{2} \left(1 - \left(\frac{|x-y|}{|x-\tilde{y}|} \right)^{2} \right) \text{ for all } x \neq y \in \mathbb{R}^N_+ \text{ and } N \geq 4,$$

we obtain the existence of a constant $c_2 > 0$ such that

$$G_{\mathbb{R}^N_+}(x,y) \leq \frac{c_2 x_N y_N}{|x-y|^{N-2}|x-\tilde{y}|^2} \quad \text{for all } x \neq y \in \mathbb{R}^N_+.$$

ii) Since $|x-y| \leq |x-\tilde{y}|$ for all $x, y \in \mathbb{R}^N_+$, **(4.5)** is a consequence of **(4.4)**.

iii) Notice that

$$\begin{aligned}
|x-\tilde{y}|^N &= |x-a_1-(\tilde{y}-a_1)|^N \\
&\leq c\left(|x-a_1|^N + |\tilde{y}-a_1|^N\right) \\
&\leq c\left((|x|+1)^N + (|y|+1)^N\right) \\
&\leq c(|x|+1)^N(|y|+1)^N,
\end{aligned}$$

for some constant $c > 0$. Therefore **(4.6)** is a consequence of **(4.5)**.

iv) Let $i = 1, \ldots, N$. Notice that

$$|x_i - y_i|\left(\frac{1}{|x-y|^N} - \frac{1}{|x-\tilde{y}|^N}\right) = c|x_i-y_i|\xi(x,y)G_{\mathbb{R}^N_+}(x,y) \quad \text{for all } x \neq y \in \mathbb{R}^N_+,$$

for some constant $c > 0$ and where we have defined

$$\xi(x,y) := \frac{|x-y|^{N-2}|x-\tilde{y}|^{N-2}}{|x-\tilde{y}|^{N-2} - |x-y|^{N-2}}\left(\frac{1}{|x-y|^N} - \frac{1}{|x-\tilde{y}|^N}\right) \quad \text{for all } x \neq y \in \mathbb{R}^N_+.$$

Thus, to prove **(4.7)**, it is enough to show that there exists $c > 0$ such that for $y \in \mathbb{R}^N_+ \backslash B(x,s)$ and each $x \neq y \in \mathbb{R}^N_+$:

$$|x_i - y_i|\xi(x,y) \leq c.$$

In fact,

$$\begin{aligned}
\xi(x,y) &= \frac{|x-\tilde{y}|^{N-2}|x-y|^{-2} - |x-y|^{N-2}|x-\tilde{y}|^{-2}}{|x-\tilde{y}|^{N-2} - |x-y|^{N-2}} \\
&= \frac{\left(|x-\tilde{y}|^{N-2} - |x-y|^{N-2}\right)\left(|x-y|^{-2} + |x-\tilde{y}|^{-2}\right) - |x-\tilde{y}|^{N-4} + |x-y|^{N-4}}{|x-\tilde{y}|^{N-2} - |x-y|^{N-2}} \\
&= |x-y|^{-2} + |x-\tilde{y}|^{-2} - \frac{|x-\tilde{y}|^{N-4} - |x-y|^{N-4}}{|x-\tilde{y}|^{N-2} - |x-y|^{N-2}} \\
&\leq |x-y|^{-2} + |x-\tilde{y}|^{-2} \quad \text{for all } x \neq y \in \mathbb{R}^N_+ \quad \text{and } N \geq 4.
\end{aligned}$$

Similarly, if $N = 3$, we have

$$\begin{aligned}
\xi(x,y) &= |x-y|^{-2} + |x-\tilde{y}|^{-2} - \frac{|x-\tilde{y}|^{-1} - |x-y|^{-1}}{|x-\tilde{y}| - |x-y|} \\
&= |x-y|^{-2} + |x-\tilde{y}|^{-2} + \frac{1}{|x-\tilde{y}||x-y|} \\
&\leq 2\left(|x-y|^{-2} + |x-\tilde{y}|^{-2}\right) \quad \text{for all } x \neq y \in \mathbb{R}^N_+.
\end{aligned}$$

Therefore

$$\xi(x,y) \le 2\left(|x-y|^{-2} + |x-\tilde{y}|^{-2}\right) \quad \text{for all } x \ne y \in \mathbb{R}^N_+ \quad \text{and } N \ge 3.$$

On the other hand, since $y \in \mathbb{R}^N_+ \backslash B(x,s)$ we also have $\tilde{y} \in \mathbb{R}^N_+ \backslash B(x,s)$ and hence

$$|x_i - y_i| \le |x-y| \le \begin{cases} \dfrac{|x-y|^2}{s} \\ \dfrac{|x-\tilde{y}|^2}{s}, \end{cases}$$

and thus

$$|x_i - y_i|\xi(x,y) \le \frac{4}{s}.$$

This completes the proof of iv).

v) Since $(x_N - y_N)^2 \le |x-y|$, we have $(x_N - y_N)^2 \le x_N y_N$. Then $x_N^2 - 3x_N y_N + y_N^2 \le 0$, or equivalently

$$\left(y_N - \left(\frac{3-\sqrt{5}}{2}\right)x_N\right)\left(y_N - \left(\frac{3+\sqrt{5}}{2}\right)x_N\right) \le 0,$$

which implies that

$$\left(\frac{3-\sqrt{5}}{2}\right)x_N \le y_N \le \left(\frac{3+\sqrt{5}}{2}\right)x_N.$$

□

The following result will help us to use the Green's formulas.

Lemma 4.0.4. *Let $\rho \in L^\infty_{loc}\left(\mathbb{R}^N_+\right)$, $\rho(x) \ge 0$ and ρ not identically zero. Assume that*

$$w(x) = \int_{\mathbb{R}^N_+} \Big(\Gamma(x-y) - \Gamma(x-\tilde{y})\Big)\rho(y)dy,$$

belongs to $L^\infty(\mathbb{R}^N_+)$. Then $w \in C^1(\mathbb{R}^N_+)$ and for any $x \in \mathbb{R}^N_+$

$$D_i w(x) = \int_{\mathbb{R}^N_+} \Big(D_i\Gamma(x-y) - D_i\Gamma(x-\tilde{y})\Big)\rho(y)dy, \quad \textit{for } i = 1,\dots,N.$$

Proof. Let $z \in \mathbb{R}^N_+$ and $\delta > 0$ such that $B(z,\delta) \Subset \mathbb{R}^N_+$. Fix $x \in B(z,\delta)$ and put $0 < \gamma < \frac{z_N - \delta}{2}$. Then it is clear that $B := B(x,\gamma) \subset \mathbb{R}^N_+$. Now, for each $i = 1,\dots,N$ we define

$$v_i(x) = \int_{\mathbb{R}^N_+} \Big(D_i\Gamma(x-y) - D_i\Gamma(x-\tilde{y})\Big)\rho(y)dy, .$$

Thus

$$v_i(x) = \frac{-1}{N w_N} \int_{\mathbb{R}^N_+} \left(\frac{x_i - y_i}{|x-y|^N} - \frac{x_i - \tilde{y}_i}{|x-\tilde{y}|^N}\right)\rho(y)dy.$$

Let $i = 1, \ldots, N-1$. Then, from **Lemma 4.0.4** iv) there exists $C > 0$ such that

$$\begin{aligned}
|v_i(x)| &= \left| \frac{-1}{nw_N} \int_{\mathbb{R}^N_+} \left(\frac{x_i - y_i}{|x-y|^N} - \frac{x_i - y_i}{|x-\tilde{y}|^N} \right) \rho(y) dy \right| \\
&= \frac{1}{nw_N} \left| \int_{\mathbb{R}^N_+ \setminus B(x,\gamma)} \left(\frac{x_i - y_i}{|x-y|^N} - \frac{x_i - y_i}{|x-\tilde{y}|^N} \right) \rho(y) dy + \int_{B(x,\gamma)} \left(\frac{x_i - y_i}{|x-y|^N} - \frac{x_i - y_i}{|x-\tilde{y}|^N} \right) \rho(y) dy \right| \\
&\le \frac{1}{Nw_N} \int_{\mathbb{R}^N_+ \setminus B(x,\gamma)} |x_i - y_i| \left(\frac{1}{|x-y|^N} - \frac{1}{|x-\tilde{y}|^N} \right) \rho(y) dy + \frac{2}{Nw_N} \int_{B(x,\gamma)} \frac{1}{|x-y|^{N-1}} \rho(y) dy \\
&\le C \int_{\mathbb{R}^N_+ \setminus B(x,\gamma)} G_{\mathbb{R}^N_+}(x,y) \rho(y) dy + \frac{2\|\rho\|_{L^\infty(B)}}{Nw_N} \int_{B(0,1)} \frac{1}{|y|^{N-1}} dy \\
&= Cw(x) + 2\|\rho\|_{L^\infty(B)},
\end{aligned}$$

If $i = N$, similar to the previous case, from **Lemma 4.0.4** ii), iv) and since x_N and y_N are less than or equal to $|x - \tilde{y}|$, there exists $C > 0$ such that

$$\begin{aligned}
|v_N(x)| &= \left| \frac{-1}{nw_N} \int_{\mathbb{R}^N_+} \left(\frac{x_N - y_N}{|x-y|^N} - \frac{x_N + y_N}{|x-\tilde{y}|^N} \right) \rho(y) dy \right| \\
&= \left| \frac{-1}{Nw_N} \int_{\mathbb{R}^N_+} \left(\frac{x_N - y_N}{|x-y|^N} - \frac{x_N - y_N}{|x-\tilde{y}|^N} \right) dx + \frac{2}{Nw_N} \int_{\mathbb{R}^N_+} \frac{y_N}{|x-\tilde{y}|^N} \rho(y) dy \right| \\
&\le Cw(x) + 2\|\rho\|_{L^\infty(B)} + \frac{2}{Nw_N} \left(\int_{\mathbb{R}^N_+ \setminus B(x,\gamma)} \frac{y_N}{|x-\tilde{y}|^N} \rho(y) dy + \int_{B(x,\gamma)} \frac{y_N}{|x-\tilde{y}|^N} \rho(y) dy \right) \\
&\le Cw(x) + 2\|\rho\|_{L^\infty(B)} + \frac{2}{Nw_N x_N} \int_{\mathbb{R}^N_+ \setminus B(x,\gamma)} \frac{x_N y_N}{|x-\tilde{y}|^N} \rho(y) dy + \frac{2\|\rho\|_{L^\infty(B)}}{Nw_N} \int_{B(x,1)} \frac{1}{|x-\tilde{y}|^{N-1}} dy \\
&\le Cw(x) + 2\|\rho\|_{L^\infty(B)} + \frac{2C}{Nw_N x_N} \int_{\mathbb{R}^N_+ \setminus B(x,\gamma)} G_{\mathbb{R}^N_+}(x,y) \rho(y) dy + \frac{2\|\rho\|_{L^\infty(B)}}{Nw_N x_N^{N-1}} \int_{B(x,1)} dy \\
&\le C \left(1 + \frac{2}{Nw_N x_N} \right) w(x) + 2 \left(1 + \frac{1}{x_N^{N-1}} \right) \|\rho\|_{L^\infty(B)}.
\end{aligned}$$

Therefore, since $w \in L^\infty(\mathbb{R}^N_+)$, it follows that v_i is well defined. We now show that $v_i = \nabla w_i$ for each $i = 1, \ldots, N$. To do so, first notice that for each $x \in \mathbb{R}^N_+$, $|x - \tilde{y}| \neq 0$, then is clear that

$$D_i \int_{\mathbb{R}^N_+} \Gamma(x - \tilde{y}) \rho(y) dy = \int_{\mathbb{R}^N_+} D_i \Gamma(x - \tilde{y}) \rho(y) dy, \quad \text{for } i = 1, \ldots, N.$$

The proof of

$$D_i \int_{\mathbb{R}^N_+} \Gamma(x - y) \rho(y) dy = \int_{\mathbb{R}^N_+} D_i \Gamma(x - y) \rho(y) dy, \quad \text{for } i = 1, \ldots, N,$$

for all $x \in \mathbb{R}^N_+$, it follows as the proof of **Lemma 2.1.4**, in which it is only necessary to eliminate the singularity $x = y$ using an auxiliary function. Therefore, $w \in C^1(\mathbb{R}^N_+)$ and $v_i = \nabla w_i$ for each $i = 1, \ldots, N$.

□

To facilitate reading we will give the definition again of the property (H$_+$), given in the introduction.

Definition 4.0.5. Let $\rho \in L^{\infty}_{loc}(\mathbb{R}^N_+)$, $\rho(x) \geq 0$ and ρ not identically zero. We say that ρ has the property (H$_+$) if there exists a bounded solution of:

$$-\Delta u = \rho(x) \text{ in } \mathbb{R}^N_+. \tag{P$_+$}$$

Next we will prove the main result of this section which gives a necessary and sufficient condition to have the property (H$_+$).

Proof of Theorem 6. Suppose the property (H$_+$) is satisfied. Then, there exists U a bounded solution of **(P$_+$)**. By adding a constant, we may always assume that $U \geq 0$ in $\mathbb{R}^N_+$. On the other hand, for each $n \in \mathbb{N}$, from **Lemma 4.0.2**, Problem **(P$_n$)** has only one increasing weak solution $u_n \in H^1_0(B_n(a_n))$. In addition

$$u_n(x) = \int_{B_n(a_n)} G_{B_n(a_n)}(x,y)\rho(y)dy. \tag{4.9}$$

Let $\varphi \in C^{\infty}_0(B_n(a_n))$ with $\varphi \geq 0$. Then, from Green's identities

$$\begin{aligned}
-\int_{B_n(a_n)} U\Delta\varphi dx &= \int_{B_n(a_n)} \nabla U \nabla \varphi dx = \int_{\mathbb{R}^N} \nabla U \nabla \varphi dx = \int_{\mathbb{R}^N} \rho(x)\varphi(x)dx \\
&\geq \int_{B_n(a_n)} \rho(x)\varphi(x)dx = \int_{B_n(a_n)} \nabla u_n \nabla \varphi dx \\
&= -\int_{B_n(a_n)} u_n \Delta\varphi dx,
\end{aligned}$$

from where

$$\int_{B_n(a_n)} (U - u_n)\Delta\varphi dx \leq 0.$$

Therefore, the maximum principle implies that $u_n \leq U$ in B_n for all $n \in \mathbb{N}$. Then

$$u(x) := \lim_{n\to\infty} u_n(x) \ \text{ exist for every } x \in \mathbb{R}^N_+,$$

and

$$u \leq U \ \text{ in } \ \mathbb{R}^N_+. \tag{4.10}$$

From representation formula **(4.9)**, using **Lemma 4.0.1** and Fatou's Lemma, we get

$$\begin{aligned}
\int_{\mathbb{R}^N_+} G_{\mathbb{R}^N_+}(x,y)\rho(y)dy &= \int_{\mathbb{R}^N_+} \Big(\Gamma(x-y)-\Gamma(x-\tilde{y})\Big)\rho(y)dy \\
&= \int_{\mathbb{R}^N_+} \lim_{n\to\infty} G_{B_n(a_n)}(x,y)\rho(y)dy \\
&= \int_{\mathbb{R}^N_+} \lim_{n\to\infty} G_{B_n(a_n)}(x,y)\chi_{B_n(a_n)}(y)\rho(y)dy \\
&\le \liminf_{n\to\infty} \int_{\mathbb{R}^N_+} G_{B_n(a_n)}(x,y)\chi_{B_n(a_n)}(y)\rho(y)dy \\
&= \liminf_{n\to\infty} \int_{B_n(a_n)} G_{B_n(a_n)}(x,y)\rho(y)dy \\
&= \liminf_{n\to\infty} u_n(x) \\
&= u(x).
\end{aligned}$$

Consequently, from **(4.10)**, we obtain

$$w_\infty(x) := \int_{\mathbb{R}^N_+} G_{\mathbb{R}^N_+}(x,y)\rho(y)dy \in L^\infty(\mathbb{R}^N_+).$$

Reciprocally, assuming $w_\infty \in L^\infty(\mathbb{R}^N_+)$, from **Lemma 4.0.4** $w_\infty \in C^1(\mathbb{R}^N_+)$. Let $\varphi \in C_0^\infty(\mathbb{R}^N_+)$, then using Green's identities, Fubini's theorem and since $\Gamma(x-\tilde{y})$ is a harmonic function for all $x, y \in \mathbb{R}^N_+$, we have

$$\begin{aligned}
\int_{\mathbb{R}^N_+} \nabla w_\infty(x)\nabla\varphi(x)dx &= -\int_{\mathbb{R}^N_+} w_\infty(x)\Delta\varphi(x)dx \\
&= -\int_{\mathbb{R}^N_+} \left(\int_{\mathbb{R}^N_+} \Big(\Gamma(x-y)-\Gamma(x-\tilde{y})\Big)\rho(y)dy\right)\Delta\varphi(x)dx \\
&= -\int_{\mathbb{R}^N_+} \rho(y)\left(\int_{\mathbb{R}^N_+} \Big(\Gamma(x-y)-\Gamma(x-\tilde{y})\Big)\Delta\varphi(x)dx\right)dy \\
&= -\int_{\mathbb{R}^N_+} \rho(y)\left(\int_{\mathbb{R}^N_+} \Gamma(x-y)\Delta\varphi(x)dx\right)dy \\
&= \int_{\mathbb{R}^N_+} \rho(y)\varphi(y)dy,
\end{aligned}$$

where the last equality is true following a similar argument to that of the proof of **Lemma 2.1.3**. Thus, the function $w_\infty \in H^1(\mathbb{R}^N_+) \cap L^\infty(\mathbb{R}^N_+)$ provides a bounded positive solution of **($\mathbf{P}_+$)**.

□

Corollary 4.0.6. *Suppose that ρ satisfies property* (H_+). *Then w_∞ is minimal positive solution of* $(\mathbf{P}_+)$.

Proof. From **Theorem 6**, since ρ satisfies property (H_+), it follows that w_∞ is a bounded positive solution of $(\mathbf{P}_+)$. Let U be a bounded positive solution of $(\mathbf{P}_+)$. The maximum principle implies implies that $u_n \le U$ in B_n for all $n \in \mathbb{N}$. Then

$$u(x) := \lim_{n\to\infty} u_n(x) \text{ exist for every } x \in \mathbb{R}^N_+,$$

and $u \le U$ in $\mathbb{R}^N_+$. Thus, using Fatou's Lemma and representation formula **(4.9)** we have

$$U(x) \ge \lim_{n\to\infty} u_n(x) \ge w_\infty(x) \text{ for every } x \in \mathbb{R}^N_+,$$

Therefore, w_∞ is minimal positive solution of $(\mathbf{P}_+)$.

□

Next, we will give some examples of ρ for which we will have existence and nonexistence of a bounded solution of the Problem $(\mathbf{P}_+)$.

Lemma 4.0.7. *Assume that $\rho(x) = 1$ for all $x \in \mathbb{R}^N_+$. Then, Problem* $(\mathbf{P}_+)$ *has no bounded solution.*

Proof. Let $x \in \mathbb{R}^N_+$ with $x_N > 1$. From **Theorem 6** and **Lemma 4.0.3** i), is sufficient to show that

$$\int_{\mathbb{R}^N_+} \frac{x_N y_N}{|x-y|^{N-2}|x-\tilde{y}|^2}\rho(y)dy = \infty.$$

In fact, notice that if $|x-y|^2 \le x_N y_N$, from **Lemma 4.0.3** v) there exists $c \in (0,1)$ such that $cx_N \le y_N$, which implies that

$$\int_{|x-y|^2\le x_N y_N} \frac{1}{|x-y|^N}dy \ge \int_{|x-y|^2 \le cx_N^2} \frac{1}{|x-y|^N}dy.$$

Hence, using

$$|x-\tilde{y}|^2 - |x-y|^2 = 4x_N y_N \text{ for all } x, y \in \mathbb{R}^N_+,$$

and considering $\mathbb{R}^N_+ \cap (|x-y|^2 \le cx_N^2) = |x-y| \le \sqrt{c}x_N$, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^N_+} \frac{x_N y_N}{|x-y|^{N-2}|x-\tilde{y}|^2}dy &\ge \int_{\mathbb{R}^N_+\cap(|x-y|^2\le x_N y_N)} \frac{x_N y_N}{|x-y|^{N-2}(|x-y|^2+4x_N y_N)}dy \\
&\ge \frac{1}{5}\int_{\mathbb{R}^N_+\cap(|x-y|^2\le cx_N^2)} \frac{1}{|x-y|^{N-2}}dy \\
&= \frac{1}{5}\int_{|x-y|\le\sqrt{c}x_N} \frac{1}{|x-y|^{N-2}}dy \\
&= \frac{Nw_N}{5}\int_0^{\sqrt{c}x_N} \frac{r^{N-1}}{r^{N-2}}dr \\
&= \frac{Nw_N}{5}\int_0^{\sqrt{c}x_N} r\,dr \\
&\to \infty \quad \text{as} \quad x_N \to \infty.
\end{aligned}$$

□

Lemma 4.0.8. *Let $\rho : \mathbb{R}^N_+ \to \mathbb{R}$ be a measurable function not identically zero that satisfies*

$$0 \le \rho(x) \le \frac{1}{(1+|x|)^\beta x_N^\gamma} \quad \textit{for } x \in \mathbb{R}^N_+,$$

with $0 \le \gamma < 1$ and $2 < \beta + \gamma$. Then, Problem ($\mathbf{P}_+$) has a solution $u \in H^1(\mathbb{R}^N_+) \cap L^\infty(\mathbb{R}^N_+)$. Furthermore if $\beta + \gamma < N + 1$, then:

$$\lim_{|x|\to\infty} u(x) = 0 \quad \textit{and} \quad \lim_{x_N \to 0} u(x) = 0.$$

Proof. Assume $0 \le \gamma < 1$ and $2 < \beta + \gamma$. From **Theorem** 6 it is sufficient to show that $w_\infty \in L^\infty(\mathbb{R}^N_+)$. For this purpose, first we will prove that

$$\int_{\mathbb{R}^N_+} \frac{x_N y_N}{|x-y|^{N-2}|x-\tilde{y}|^2} \rho(y) dy \in L^\infty(\mathbb{R}^N_+), \tag{4.11}$$

since **Lemma** 4.0.3 i) would implies that $w_\infty \in L^\infty(\mathbb{R}^N_+)$.

We estimate the previous integral by separating the half space as the union of $\mathbb{R}^N_+ \cap B(x,1)$ with $\mathbb{R}^N_+ \backslash B(x,1)$, where $x \in \mathbb{R}^N_+$.

In fact, using that $x_N \le |x - \tilde{y}|$ and $y_N \le |x-\tilde{y}|$ we have

$$\begin{aligned}
\int_{\mathbb{R}^N_+ \cap B(x,1)} \frac{x_N y_N}{|x-y|^{N-2}|x-\tilde{y}|^2} \rho(y) dy &\le \int_{B(x,1)} \frac{y_N^{1-\gamma}}{|x-y|^{N-2}|x-\tilde{y}|} dy \\
&= \int_{B(x,1)} \frac{y_N^{1-\gamma}}{|x-y|^{N-2}|x-\tilde{y}|^{1-\gamma}|x-\tilde{y}|^{\gamma}} dy \\
&\le \int_{B(x,1)} \frac{1}{|x-y|^{N+\gamma-2}} dy \\
&= N w_N \int_0^1 \frac{r^{N-1}}{r^{N+\gamma-2}} dr \\
&= N w_N \int_0^1 r^{1-\gamma} dr \\
&= \frac{N w_N}{2-\gamma}.
\end{aligned}$$

Now, to estimate (4.11) in $\mathbb{R}^N_+ \backslash B(x,1)$ we will separate in two cases:

i) Assume that $|x| < 1$. Since for $y \in \mathbb{R}^N_+ \backslash B(x,1)$ we have

$$|y| = |x-y-x| \ge \Big| |x-y| - |x| \Big| = |x-y| - |x| \ge |x-y| - 1,$$

then

$$\begin{aligned}\int_{\mathbb{R}^N_+\setminus B(x,1)} \frac{x_N y_N}{|x-y|^{N-2}|x-\tilde{y}|^2}\rho(y)dy &\leq \int_{|x-y|\geq 1} \frac{1}{|x-y|^{N+\gamma-2}(1+|y|)^\beta}dy \\ &\leq \int_{|x-y|\geq 1} \frac{1}{|x-y|^{N+\beta+\gamma-2}}dy \\ &= Nw_N \int_1^\infty r^{1-\beta-\gamma}dr \\ &= \frac{Nw_N}{\beta+\gamma-2}.\end{aligned}$$

ii) Assume that $|x| \geq 1$. To estimate

$$\int_{\mathbb{R}^N_+\setminus B(x,1)} \frac{x_N y_N}{|x-y|^{N-2}|x-\tilde{y}|^2}\rho(y)dy \tag{4.12}$$

let us first notice that

$$|x-y| \geq \Big||x|-|y|\Big| = |y|-|x| \geq \frac{|y|}{2},$$

provided that $|y| \geq 2|x|$. This implies:

$$\begin{aligned}\int_{(\mathbb{R}^N_+\setminus B(x,1))\cap(|y|\geq 2|x|)} \frac{x_N y_N}{|x-y|^{N-2}|x-\tilde{y}|^2}\rho(y)dy &\leq \int_{|y|\geq 2|x|} \frac{|x|y_N^{1-\gamma}}{|x-y|^N|y|^\beta}dy \\ &\leq \frac{1}{2}\int_{|y|\geq 2|x|} \frac{|y|^{2-\gamma}}{|x-y|^N|y|^\beta}dy \\ &\leq 2^{N-1}\int_{|y|\geq 2|x|} \frac{1}{|y|^{N+\beta+\gamma-2}}dy \\ &= 2^{N-1}Nw_N\int_{2|x|}^\infty r^{1-\beta-\gamma}dr \\ &= \frac{2^{N-\beta-\gamma}Nw_N}{\beta+\gamma-2}|x|^{2-\beta-\gamma} \\ &\leq \frac{2^{N-\beta-\gamma}Nw_N}{\beta+\gamma-2}.\end{aligned}$$

Now, to estimate **(4.12)** in $\mathbb{R}^N_+\setminus B(x,1))\cap(|y|\leq 2|x|)$, we will do it in 3 regions:

$$A_1 = (\mathbb{R}^N_+\setminus B(x,1))\cap\left(\frac{|x|}{2}\leq |y| \leq 2|x|\right),$$

and

$$A_2 = (\mathbb{R}^N_+\setminus B(x,1))\cap\left(\frac{1}{2}\leq |y| \leq \frac{|x|}{2}\right), A_3 = (\mathbb{R}^N_+\setminus B(x,1))\cap\left(|y|\leq \frac{1}{2}\right).$$

The first one:

$$\int_{A_1} \frac{x_N y_N}{|x-y|^{N-2}|x-\tilde{y}|^2}\rho(y)dy \leq \int_{(\mathbb{R}^N_+\setminus B(x,1))\cap(\frac{|x|}{2}\leq|y|\leq 2|x|)} \frac{1}{|x-y|^{N+\gamma-2}|y|^\beta}dy$$

$$\leq \frac{2^\beta}{|x|^\beta}\int_{1\leq|x-y|\leq 3|x|}\frac{1}{|x-y|^{N+\gamma-2}}dy$$

$$= \frac{2^\beta N w_N}{|x|^\beta}\int_1^{3|x|} r^{1-\gamma}dr$$

$$\leq \frac{2^\beta 3^{2-\gamma} N w_N}{2-\gamma}|x|^{2-\beta-\gamma}$$

$$\leq \frac{2^\beta 3^{2-\gamma} N w_N}{2-\gamma}.$$

To estimate **(4.12)** in the second region, note that if $|y| \leq \frac{|x|}{2}$, we have

$$|x-y| \geq \Big||x|-|y|\Big| = |x|-|y| \geq \frac{|x|}{2} \geq |y|,$$

then

$$\int_{A_2} \frac{x_N y_N}{|x-y|^{N-2}|x-\tilde{y}|^2}\rho(y)dy \leq \int_{(\mathbb{R}^N_+\setminus B(x,1))\cap(\frac{1}{2}\leq|y|\leq \frac{|x|}{2})} \frac{1}{|x-y|^{N+\gamma-2}|y|^\beta}dy$$

$$\leq \int_{(\mathbb{R}^N_+\setminus B(x,1))\cap(\frac{1}{2}\leq|y|\leq \frac{|x|}{2})} \frac{1}{|y|^{N+\beta+\gamma-2}}dy$$

$$\leq \int_{\frac{1}{2}\leq|y|} \frac{1}{|y|^{N+\beta+\gamma-2}}dy$$

$$= N w_N \int_{\frac{1}{2}}^{\infty} r^{1-\beta-\gamma}dr$$

$$= \frac{N w_N}{(\beta+\gamma-2)2^{2-\beta-\gamma}}.$$

Finally, to estimate **(4.12)** in the third region, notice that if $|y| \leq \frac{1}{2}$, we have

$$|x-y| \geq \Big||x|-|y|\Big| = |x|-|y| \geq |x|-\frac{1}{2},$$

then

$$\int_{A_3} \frac{x_N y_N}{|x-y|^{N-2}|x-\tilde{y}|^2}\rho(y)dy \leq \int_{(\mathbb{R}^N_+\setminus B(x,1))\cap(|y|\leq\frac{1}{2})} \frac{1}{|x-y|^{N+\gamma-2}}dy$$

$$\leq \frac{1}{(|x|-\frac{1}{2})^{N+\gamma-2}}\int_{|y|\leq\frac{1}{2}} dy$$

$$= \frac{w_N}{2^N(|x|-\frac{1}{2})^{N+\gamma-2}}$$

$$\leq w_N 2^{\gamma-2}.$$

Therefore, there exists $C > 0$ such that

$$\int_{\mathbb{R}^N_+ \setminus B(x,1)} \frac{x_N y_N}{|x-y|^{N-2}|x-\tilde{y}|^2} \rho(y) dy \leq C.$$

Hence, we get existence of a constant $C > 0$ such that

$$\int_{\mathbb{R}^N_+} \frac{x_N y_N}{|x-y|^{N-2}|x-\tilde{y}|^2} \rho(y) dy \leq C \quad \text{for all } x \in \mathbb{R}^N_+.$$

Now, we assume $0 \leq \gamma < 1$ and $2 < \beta + \gamma < N + 1$. Following an argument similar to the previous one, it is possible to show that

$$\lim_{x_N \to 0} w_\infty(x) = 0.$$

For instance if $|x| < 1$, it follows that

$$\begin{aligned} \int_{\mathbb{R}^N_+ \setminus B(x,1)} \frac{x_N y_N}{|x-y|^{N-2}|x-\tilde{y}|^2} \rho(y) dy &\leq x_N \int_{|x-y| \geq 1} \frac{1}{|x-y|^{N+\gamma-1}(1+|y|)^\beta} dy \\ &\leq x_N \int_{|x-y| \geq 1} \frac{1}{|x-y|^{N+\beta+\gamma-1}} dy \\ &= \frac{N w_N}{\beta + \gamma - 1} x_N, \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^N_+ \cap B(x,1)} \frac{x_N y_N}{|x-y|^{N-2}|x-\tilde{y}|^2} \rho(y) dy &\leq x_N \int_{B(x,1)} \frac{1}{|x-y|^{N+\gamma-1}} dy \\ &= N w_N x_N \int_0^1 r^{-\gamma} dr \\ &= \frac{N w_N}{1-\gamma} x_N. \end{aligned}$$

Now, to prove

$$\lim_{|x| \to \infty} w_\infty(x) = 0,$$

we see that it is enough to show that

$$\int_{(\mathbb{R}^N_+ \setminus B(x,\varepsilon)) \cap (|y| \leq \frac{|x|}{2})} \frac{x_N y_N}{|x-y|^{N-2}|x-\tilde{y}|^2} \rho(y) dy \to 0 \text{ as } |x| \to \infty, \tag{4.13}$$

for $\varepsilon > 0$ enough small.

Finally we will prove (4.13). In fact, using that $|x-y| \geq \frac{|x|}{2}$ and since $N-\beta-\gamma+1>0$, it follows that

$$\begin{aligned}
\int_{(\mathbb{R}^N_+ \setminus B(x,\varepsilon)) \cap (|y| \leq \frac{|x|}{2})} \frac{x_N y_N}{|x-y|^{N-2}|x-\tilde{y}|^2}\rho(y)dy &\leq \int_{|y| \leq \frac{|x|}{2}} \frac{|x| y_N^{1-\gamma}}{|x-y|^N |y|^\beta} dy \\
&\leq \frac{2^N}{|x|^{N-1}} \int_{|y| \leq \frac{|x|}{2}} \frac{1}{|y|^{\beta+\gamma-1}} dy \\
&= \frac{2^N N w_N}{|x|^{N-1}} \int_0^{\frac{|x|}{2}} r^{N-\beta-\gamma} dr \\
&= \frac{2^{\beta+\gamma-1} N w_N}{N-\beta-\gamma+1} |x|^{2-\beta-\gamma}.
\end{aligned}$$

This concludes the proof.

□

To finish, we have the following nonexistence result.

Lemma 4.0.9. *Let $\rho : \mathbb{R}^N_+ \to \mathbb{R}$ be a measurable function that belongs to $L^\infty_{loc}\left(\mathbb{R}^N_+\right)$ and satisfies*

$$\frac{1}{(1+|x|)^\beta x_N^\gamma} \leq \rho(x) \quad \text{for } x \in \mathbb{R}^N_+,$$

with $\gamma \geq 1$ and $\beta+\gamma \leq 2$. Then, Problem ($\mathbf{P}_+$) *has no bounded solution.*

Proof. To prove that the Problem ($\mathbf{P}_+$) has no bounded solution, from **Theorem 6** and **Lemma 4.0.3** i), it is sufficient to show that

$$\int_{\mathbb{R}^N_+ \cap (|y| \geq |x|)} \frac{x_N y_N}{|x-\tilde{y}|^N} \rho(y) dy = \infty.$$

In fact, let $x \in \mathbb{R}^N_+$ with $x_N > 2^{N+1}$. Using that $|x-\tilde{y}|^N \leq 2^N\left(|x|^N + |\tilde{y}|^N\right) \leq 2^{N+1}|y|^N$ for $|y| \geq |x|$, we have

$$\frac{x_N}{|x-\tilde{y}|^N} \geq \frac{1}{|y|^N}.$$

Thus, there exists $C>0$ such that

$$\begin{aligned}
\int_{\mathbb{R}^N_+ \cap (|y| \geq |x|)} \frac{x_N y_N}{|x-\tilde{y}|^N} \rho(y) dy &\geq \int_{\mathbb{R}^N_+ \cap |y| \geq |x|} \frac{1}{|y|^N (1+|y|)^\beta y_N^{\gamma-1}} dy \\
&\geq C \int_{|y| \geq |x|} \frac{1}{|y|^{N+\beta+\gamma-2}} dy, \\
&= \infty
\end{aligned}$$

which implies the desired. □

Remark 4.0.1. From **Theorem 6**, if ρ satisfies property (H_+), then w_∞ is a bounded positive solution of Problem ($\mathbf{P}_+$). Furthermore, since for the problem in the whole space $\mathbb{R}^N$, we have show that the solution u_∞ vanishing at infinity (see **Corollary 2.1.9**), we will also hope that

$$\liminf_{|x|\to\infty} w_\infty(x) = 0 \quad \text{and} \quad \liminf_{x_N \to 0} w_\infty(x) = 0.$$

However, this claim still is open problem.

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